An Axiomatic Approach to Semiclassical Field Perturbation Theory

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Abstract

Semiclassical perturbation theory is investigated within the framework of axiomatic field theory. Axioms of perturbation semiclassical theory are formulated. Their correspondence with LSZ approach and Schwinger source theory is studied. Semiclassical S-matrix, as well as examples of decay processes, are considered in this framework.

Keywords: Maslov semiclassical theory, axiomatic quantum field theory, Bogoliubov S-matrix, Lehmann-Symanzik-Zimmermann approach, Schwinger sources, Peierls brackets.

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1 Introduction

The main difficulty of quantum field theory (QFT) is that there is no nontrivial model satisfying all the axioms. In fact, QFT is constructed within the perturbation theory framework.

However, the perturbation theory is a partial case of the semiclassical theory. Therefore, it is useful to generalize an axiomatic approach to perturbation QFT to the semiclassical theory. This is the problem to be considered in this paper. The ideas of [1] are developed here.

General structure of semicalssical perturbation theory in QFT is investigated in section 2. The main object of semiclassical theory is *a semiclassical bundle* [2]. Points on the space of the bundle are interpreted as possible semiclassical states. The base of the bundle is the classical state space, fibres are spaces of quantum states in a given external classical background.

In addition to the "point-type" states, one can also consider their superpositions.

Important objects are introduced in QFT. These are Poincare transformation unitary operators $\mathcal{U}_g^h$ and Heisenberg field operators. There analogs should arise in the semiclassical theory as well. These semiclassical structures are also discussed in section 2.

Section 3 deals with the specific features of the covariant approach to the semiclassical field theory. Its relationship with the axiomatic field theory [3], Schwinger source theory [4], LSZ approach [5], S-matrix Bogoliubov theory [6, 7].

Section 4 is devoted to the leading order of the semiclassical theory. All axioms of the semiclassical field theory are checked.

The semiclassical perturbation theory is discussed in section 5. The calculations can be simplified, provided that the asymptotic condition of the S-matrix approach [3, 6, 7, 8] is satisfied.

It is well-known that there are difficulties of the S-matrix approach due to unstable particles and bound states [3]. In section 6 we show how one can develop the semiclassical perturbation theory with unstable particles. An example of particle decay is considered.

Section 7 contains the concluding remarks.

2 General structure of semiclassical perturbation theory in QFT

2.1. We consider the quantum field system with the Lagrangian that depends on the

small parameter h as follows:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{h}V(\sqrt{h}\varphi). \tag{2.1}$$

Different methods to develop the semiclassical approximation theory for this model are known. One can use both Hamiltonian and manifestly covariant approaches. One considers "semiclassical" states that depend on h as $h \to 0$ as follows (see [9], [10] and references therein). For the Hamiltonian approach,

$$\Psi \simeq e^{\frac{i}{h}S} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}[\Pi(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \Phi(\mathbf{x})\hat{\pi}(\mathbf{x})]} f \equiv K^h_{S,\Pi,\Phi} f, \tag{2.2}$$

Here $\hat{\varphi}$ and $\hat{\pi}$ are field and momenta operators, f is a state vector that expands into a series in $\sqrt{h}$. For the manifestly covariant approach,

$$\Psi \simeq e^{\frac{i}{h}\overline{S}} T e^{\frac{i}{\sqrt{h}}\int dx J(x)\hat{\varphi}(x)} \overline{f} \equiv e^{\frac{i}{h}\overline{S}} T^h_J \overline{f} \equiv K^h_{\overline{S},J} \overline{f}. \tag{2.3}$$

Here $\hat{\varphi}(x)$ is a Heisenberg field, $J(x)$ is a classical source with a compact source, $\overline{f}$ is expanded in $\sqrt{h}$. A semiclassical state of the form (2.2) (or (2.3)) can be viewed as a point on the space of the semiclassical bundle [2]. Base of the bundle is set $\{X = (S, \Pi(\mathbf{x}), \Phi(\mathbf{x}))\}$ (or $\{\overline{X} = (\overline{S}, J(x))\}$) of classical states, fibers are spaces $\{f\}$ $(\{\overline{f}\})$ of quantum states in the external field. States (2.2) and (2.3) are written as $K^h_X f$; the operator K^h_X is called as a canonical operator.

The Maslov theory of Lagrangian manifolds with complex germs [11, 12] is a generalization of the Maslov complex germ theory. One considers states of the more complicated form:

$$\int d\alpha K^h_{X(\alpha)} f(\alpha), \quad \alpha = (\alpha_1, ..., \alpha_k), \tag{2.4}$$

They can be interpreted as k-dimensional surface on the semiclassical bundle.

Within the semiclassical theory, one can formulate the following problems:

- let $\mathcal{U}^h_g$ be a Poincare transformation corresponding to an element g of the Poincare group G, $\hat{\varphi}(x)$ be a Heisenberg field; one should investigate these operators as $h \to 0$;

- one should find norm of the state (2.4) as $h \to 0$;

- one should investigate whether the expressions (2.3) corresponding to different sources J may approximately coincide as $h \to 0$.

2.2. It happens that the following commutation rules are satisfied:

$$\begin{aligned} &\mathcal{U}^h_g K^h_X f = K^h_{u_g X} \underline{U}_g(u_g X \leftarrow X) f; \\ &\sqrt{h}\hat{\varphi}(x) K^h_X f = K^h_X \underline{\Phi}(x|X) f, \end{aligned} \tag{2.5}$$

Here $\underline{U}_g(u_gX \leftarrow X)$ is an unitary operator which is expanded into an asymptotic series in $\sqrt{h}$, u_gX is a Poincare transformation of classical state, $\underline{\Phi}(x|X)$ is an operator-valued distribution. It is also presented as an asymptotic series in $\sqrt{h}$. Its leading order is a c-number quantity $\Phi(x|X)$:

$$\underline{\Phi}(x|X) = \Phi(x|X) + \sqrt{h}\Phi^{(1)}(x|X) + ... \tag{2.6}$$

Investigate the properties of the introduced objects. Since the operators $\mathcal{U}_g^h$ should obey the group property

$$\mathcal{U}^h_{g_1g_2} = \mathcal{U}_{g_1}\mathcal{U}_{g_2},$$

it follows from eq.(2.5) that

$$\begin{gathered} u_{g_1g_2} = u_{g_1}u_{g_2}; \\ \underline{U}_{g_1g_2}(u_{g_1g_2}X \leftarrow X) = \underline{U}_{g_1}(u_{g_1g_2}X \leftarrow u_{g_2}X)\underline{U}_{g_2}(u_{g_2}X \leftarrow X). \end{gathered} \tag{2.7}$$

Moreover, the Poincare covariance of the fields implies that

$$\mathcal{U}^h_{g^{-1}}\hat{\varphi}(x)\mathcal{U}^h_g = \hat{\varphi}(w_gx), \qquad w_gx = \Lambda^{-1}(x-a) \tag{2.8}$$

Therefore, one finds from eq.(2.5) that

$$\underline{\Phi}(x|u_gX)\underline{U}_g(u_gX \leftarrow X) = \underline{U}_g(u_gX \leftarrow X)\underline{\Phi}(w_gx|X). \tag{2.9}$$

2.3. Let us estimate the square of the norm of state (2.4) as $h \to 0$. The plan is as follows (cf. [13]). The integral

$$\int d\alpha d\alpha'(K^h_{X(\alpha)}f(\alpha), K^h_{X(\alpha')}f(\alpha')) \tag{2.10}$$

is calculated with the help of the substitution $\alpha' = \alpha + \sqrt{h}\beta$. Then one performs an expansion in $\sqrt{h}$. To do this, it is necessary to use the formula of expansion of the vector $K^h_{X(\alpha+\beta\sqrt{h})}f(\alpha+\beta\sqrt{h})$ in $\sqrt{h}$.

One can obtain this formula from the commutation rule:

$$ih\frac{\partial}{\partial\alpha_a}K^h_{X(\alpha)} = K^h_{X(\alpha)}\underline{\omega}_{X(\alpha)}[\frac{\partial X}{\partial \alpha_a}]. \tag{2.11}$$

Here $\omega_X[\delta X]$ is an operator-valued 1-form. It assins an operator in $\{f\}$ to each tangent vector δX to the base. In the leading order, the 1-form is a c-number:

$$\underline{\omega}_X[\delta X] = \omega_X[\delta X] + \sqrt{h}\omega^{(1)}_X[\delta X] + ...$$

Property $[ih\frac{\partial}{\partial\alpha_a}, ih\frac{\partial}{\partial\alpha_b}] = 0$ implies the commutation relation for 1-forms:

$$\left[\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]; \omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_b}]\right] = -ih\left[\frac{\partial}{\partial\alpha_a}\underline{\omega}_{X(\alpha)}[\frac{\partial X}{\partial\alpha_b}] - \frac{\partial}{\partial\alpha_b}\underline{\omega}_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]\right], \tag{2.12}$$

or

$$[\underline{\omega}_X[\delta X_1], \underline{\omega}_X[\delta X_2]] = -ihd\underline{\omega}_X(\delta X_1, \delta X_2).$$

The operator $K^h_{X(\alpha+\sqrt{h}\beta)}$ can be expanded then in $\sqrt{h}$ as follows. Set

$$K^h_{X(\alpha+\sqrt{h}\beta)} = K^h_{X(\alpha)}V_h(\alpha,\beta). \tag{2.13}$$

Differentiate (2.13) with respect to β_a. Making use of eq.(2.11), we obtain

$$\frac{\partial}{\partial\beta_a}V_h(\alpha,\beta) = -\frac{i}{\sqrt{h}}V_h(\alpha,\beta)\underline{\omega}_{X(\alpha+\beta\sqrt{h})}[\frac{\partial X}{\partial\alpha_a}(\alpha+\beta\sqrt{h})]. \tag{2.14}$$

Therefore, the leading order in h gives us a relation

$$V_h(\alpha,\beta) \sim e^{-\frac{i}{\sqrt{h}}\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]\beta_a}.$$

The inner prooduct (2.10) is taken to the form

$$h^{k/2}\int d\alpha d\beta (f(\alpha), V_h(\alpha,\beta)f(\alpha+\beta\sqrt{h})); \tag{2.15}$$

The integrand is a rapidly oscillating quantity. Therefore, the integral will be exponentially small, except for the special case when *the Maslov isotropic condition* is satisfied:

$$\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}] = 0. \tag{2.16}$$

For the case (2.16), the system (2.14) can be solved within the perturbation framework, iff the self-consistent condition (2.12) is satisfied. For the leading order, one has

$$V_h(\alpha,\beta) \simeq e^{-i\omega^{(1)}_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]\beta_a},$$

The inner product (2.15) takes the form

$$h^{k/2}\int d\alpha d\beta (f(\alpha), \prod_a\{2\pi\delta(\omega^{(1)}_X[\frac{\partial X}{\partial\alpha_a}])\}f(\alpha)).$$

Notice that eqs.(2.5) and (2.11) imply the relations

$$\begin{gathered}\underline{U}_g(u_gX \leftarrow X)\underline{\omega}_X[\tfrac{\partial X}{\partial \alpha_a}] = \underline{\omega}_{u_gX}[\tfrac{\partial (u_gX)}{\partial \alpha_a}]\underline{U}_g(u_gX \leftarrow X) + ih\tfrac{\partial}{\partial \alpha_a}\underline{U}_g(u_gX \leftarrow X); \\ ih\tfrac{\partial}{\partial \alpha_a}\underline{\Phi}(x|X) = \left[\underline{\Phi}(x|X); \underline{\omega}_X[\tfrac{\partial X}{\partial \alpha_a}]\right].\end{gathered} \tag{2.17}$$

2.4. In the covariant framework, some of states (2.3) approximately coincide as $h \to 0$. This means that one sould introduce an equivalence relation of the base of the semiclassical bundle (on the classical state space). Some of classical states are equivalent. Moreover, if $X_1 \sim X_2$, then

$$K^h_{X_1}\overline{f}_1 \simeq K^h_{X_2}\overline{f}_2 \tag{2.18}$$

iff

$$\overline{f}_2 = \underline{V}(X_2 \leftarrow X_1)\overline{f}_1.$$

Investigate the properties of the operator $\underline{V}(X_2 \leftarrow X_1)$. First of all, the following relation

$$\underline{V}(X_3 \leftarrow X_1) = \underline{V}(X_3 \leftarrow X_2)\underline{V}(X_2 \leftarrow X_1) \tag{2.19}$$

should be satisfied. Moreover, eq. (2.18) implies that $\mathcal{U}^h_g K^h_{X_1}\overline{f}_1 \simeq \mathcal{U}^h_g K^h_{X_2}\overline{f}_2$; therefore

$$\underline{V}(u_gX_2 \leftarrow u_gX_1)\underline{U}_g(u_gX_1 \leftarrow X_1) = \underline{U}_g(u_gX_2 \leftarrow X_2)\underline{V}(X_2 \leftarrow X_1). \tag{2.20}$$

It follows from the relation $\sqrt{h}\hat{\varphi}(x)K^h_{X_1}\overline{f}_1 \simeq \sqrt{h}\hat{\varphi}(x)K^h_{X_2}\overline{f}_2$ that

$$\underline{\Phi}(x|X_2)\underline{V}(X_2 \leftarrow X_1) = \underline{V}(X_2 \leftarrow X_1)\underline{\Phi}(x|X_1). \tag{2.21}$$

Finally, let $(X_i, \overline{f}_i)$ depend on α. Differentiate (2.18) with respect to α_a: $ih\frac{\partial}{\partial \alpha_a}K^h_{X_1}\overline{f}_1 \simeq ih\frac{\partial}{\partial \alpha_a}K^h_{X_2}\overline{f}_2$; therefore,

$$\underline{V}(X_2 \leftarrow X_1)\underline{\omega}_{X_1}[\frac{\partial X_1}{\partial \alpha_a}] = \underline{\omega}_{X_2}[\frac{\partial X_2}{\partial \alpha_a}]\underline{V}(X_2 \leftarrow X_1) + ih\frac{\partial}{\partial \alpha_a}\underline{V}(X_2 \leftarrow X_1) \tag{2.22}$$

provided that $X_1(\alpha) \sim X_2(\alpha)$.

2.5. Thus, all the problems of semiclassical theory can be solved within the perturbation framework iff one specifies:

- the Poincare transformations u_g (classical) $\underline{U}_g(u_gX \leftarrow X)$ (unitary operator expanded into a formal series in $\sqrt{h}$);

- semiclassical series in $\sqrt{h}$ for $\underline{\Phi}(x|X)$ $\underline{\omega}_X[\delta X]$ (these opeators are c-numbers in the leading order);

- semiclassical series in $\sqrt{h}$ for the operators $\underline{V}(X_2 \leftarrow X_1)$ as $X_1 \sim X_2$ (if the equivalence relation is introduced on the classical state space);

These objects should satisfy the properties (2.7), (2.9), (2.12), (2.17), (2.19), (2.20), (2.21), (2.22).

Therefore, let us say that a model of semiclassical field theory is *given* iff the objects u_g, $\underline{U}_g(u_g X \leftarrow X)$, $\underline{\Phi}(x|X)$, $\underline{\omega}_X[\delta X]$, $\underline{V}(X_2 \leftarrow X_1)$ are specified and they obey the required properties. We suppose the semiclassical model to be well-defined, *even if the corresponding exact QFT model is ill-defined.*

3 Specific features of the covariant approach to semiclassical perturbation theory

3.1. Let us investigate the objects arising in the covariant approach to semiclassical field theory. First, notice that eq. (2.8) implies that

$$\mathcal{U}_g^h T_J^h \overline{f} = T_{u_g J}^h \mathcal{U}_g^h \overline{f},$$

with $u_g J(x) = J(w_g x)$, w_g being of the form (2.8). Therefore, an explicit form of transformation u_g is known, the property $u_{g_1 g_2} = u_{g_1} u_{g_2}$ is satisifed, while the operator $\underline{U}_g(u_g X \leftarrow X) \equiv \underline{U}_g = \mathcal{U}_g^h$ is X-independent and satisfies the group property and property of invariance of the fields.

$$\underline{U}_{g_1 g_2} = \underline{U}_{g_1} \underline{U}_{g_2}, \quad \underline{U}_{g^{-1}} \hat{\varphi}(x) \underline{U}_g = \hat{\varphi}(w_g x). \tag{3.1}$$

The 1-form $\underline{\omega}$ can be expressed via the LSZ R-functions [5]:

$$R(x|J) \equiv -ih (T_J^h)^+ \frac{\delta T_J^h}{\delta J(x)}. \tag{3.2}$$

Namely,

$$ih \delta K_{\overline{S},J}^h = K_{\overline{S},J}^h [-\delta \overline{S} - \int dx R(x|J) \delta J(x)],$$

therefore,

$$\underline{\omega}_X[\delta X] = -\delta \overline{S} - \int dx R(x|J) \delta J(x), \tag{3.3}$$

Moreover, $R(x|J)$ should be expanded into a formal series in $\sqrt{h}$

$$R(x|J) = \overline{\Phi}(x|J) + \sqrt{h} R^{(1)}(x|J) + ...$$

The c-number function $\overline{\Phi}(x|J)$ is called as a *classical field generated by the source* J.

Eq.(3.2) implies the following properties of the Hermitian R-function:

- Poincare invariance

$$\underline{U}_{g^{-1}} R(x|u_g J) \underline{U}_g = R(w_g x|J); \tag{3.4}$$

- Bogoliubov causality property:

$$\frac{\delta R(x|J)}{\delta J(y)} = 0, \qquad y \underset{\sim}{>} x. \tag{3.5}$$

- commutation relation

$$[R(x|J); R(y|J)] = -ih \left(\frac{\delta R(x|J)}{\delta J(y)} - \frac{\delta R(y|J)}{\delta J(x)} \right); \tag{3.6}$$

- boundary condition

$$R(x|J) = \hat{\varphi}(x)\sqrt{h}, \qquad x \underset{\sim}{<} suppJ. \tag{3.7}$$

The operator $\underline{\Phi}(x|J)$ can be expressed via the R-function at $x \underset{\sim}{>} suppJ$:

$$\underline{\Phi}(x|X) = R(x|J), \qquad x \underset{\sim}{>} suppJ. \tag{3.8}$$

3.2. Investigate the equivalence property. Say that $J \sim 0$ iff

$$T_J^h \overline{f} \simeq e^{\frac{i}{h}\overline{I}_J} \underline{W}_J \overline{f} \tag{3.9}$$

for some c-number phase $\overline{I}_J$ and operator $\underline{W}_J$ presented as a formal perturbation series in $\sqrt{h}$. The following properties are satisfied:

- relativistic invariance:

$$\underline{U}_g \underline{W}_J \underline{U}_{g^{-1}} = \underline{W}_{u_g J}, \qquad \overline{I}_{u_g J} = \overline{I}_J; \tag{3.10}$$

- unitarity

$$\underline{W}_J^+ = \underline{W}_J^{-1}; \tag{3.11}$$

- Bogoliubov causality: if $J + \Delta J_2 \sim 0$, $J + \Delta J_1 + \Delta J_2 \sim 0$ and $supp\Delta J_2 \underset{\sim}{>} supp\Delta J_1$ then the operator $(\underline{W}_{J+\Delta J_2})^+ \underline{W}_{J+\Delta J_1+\Delta J_2}$ and c-number $-\overline{I}_{J+\Delta J_2} + \overline{I}_{J+\Delta J_1+\Delta J_2}$ do not depend on ΔJ_2;

- variational property:

$$\delta \overline{I}_J - ih \underline{W}_J^+ \delta \underline{W}_J = \int dx R(x|J) \delta J(x); \tag{3.12}$$

- boundary condition

$$R(x|J) = \underline{W}_J^+ \hat{\varphi}(x) \sqrt{h} \underline{W}_J, \quad x \underset{\sim}{>} suppJ. \tag{3.13}$$

It follows from eq.(3.13) that $\overline{\Phi}(x|J) = 0$ as $x \underset{\sim}{>} suppJ$. Therefore, the classical field generated by the source $J \sim 0$ has a compact support. The following requirement allows us to construct the covariant semiclassical field theory without additional postulating equations of motion and commutation relations. Namely, suppose that *for any field configuration $\Phi(x)$ with compact support there exists a source $J \sim 0$ (denoted as $J = J_\Phi = J(x|\Phi)$) that generates the configuration Φ: $\Phi(x) = \overline{\Phi}(x|J)$. It satisfies the locality property: $\frac{\delta J(x|\Phi)}{\delta \Phi(y)} = 0$ for $x \neq y$.*

Eq.(3.12) implies in the leading order in h that the functional

$$I[\Phi] = \overline{I}_{J_\Phi} - \int dx J_\Phi(x) \Phi(x) \tag{3.14}$$

obeys the relation

$$J(x) = - \frac{\delta I[\Phi]}{\delta \Phi(x)}. \tag{3.15}$$

It is a classical equation of motion. The functional $I[\Phi]$ satisfying the locality property

$$\frac{\delta^2 I}{\delta \Phi(x) \delta \Phi(y)} = 0, \quad x \neq y \tag{3.16}$$

will be called as a *classical action of the theory*.

Denote $\underline{W}[\overline{\Phi}] \equiv \underline{W}_{J_\Phi}$. The obtained relations can be formulated as follows:

- relativistic invariance

$$\underline{U}_g \underline{W}[\overline{\Phi}] \underline{U}_{g^{-1}} = \underline{W}[u_g \overline{\Phi}]; \tag{3.17}$$

- unitarity

$$\underline{W}^+[\overline{\Phi}] = (\underline{W}[\overline{\Phi}])^{-1}; \tag{3.18}$$

- Bogoliubov causality

$$\frac{\delta}{\delta \overline{\Phi}(y)} \left(\underline{W}^+[\overline{\Phi}] \frac{\delta \underline{W}[\overline{\Phi}]}{\delta \overline{\Phi}(x)} \right) = 0, \quad y \underset{\sim}{>} x; \tag{3.19}$$

- the Yang-Feldman relation:

$$\int dy \frac{\delta^2 I}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)}[R(y|J) - \overline{\Phi}(y|J)] = ih\underline{W}^+[\overline{\Phi}]\frac{\delta \underline{W}[\overline{\Phi}]}{\delta\overline{\Phi}(x)}; \quad (3.20)$$

- the boundary condition

$$\underline{W}^+[\overline{\Phi}]\hat{\varphi}(x)\sqrt{h}\underline{W}[\overline{\Phi}] = R(x|J), \quad x \underset{\sim}{>} supp\overline{\Phi}. \quad (3.21)$$

Differential form of the Bogoliubov causality property (3.19) is obtained by a standard procedure. The Yang-Feldman relation (3.20) is derived from the variational property (3.12) with the help of the substitution

$$\delta J(x) = -\int dy \frac{\delta^2 I}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)}\delta\overline{\Phi}(y).$$

3.3. It happens that all objects of the semiclassical field theory considered in the previous subsection can be reconstructed if one specifies:

- action $I[\Phi]$ satisfying the locality and Poincare invariance property;
- operators $\hat{\varphi}(x)$ and $\underline{U}_g$ expanded in $\sqrt{h}$ and satisfying the properties (3.1);
- Hermitian operators $R(x|J)$ expanded in $\sqrt{h}$ and satisfying the properties (3.4), (3.5), (3.6), (3.7); in the leading order, the operators $R(x|J)$ should be equal to the solution $\overline{\Phi}(x|J)$ of eq.(3.15) under condition $\overline{\Phi}|_{x \underset{\sim}{<} suppJ} = 0$;
- operator $\underline{W}[\overline{\Phi}]$ expanded in $\sqrt{h}$ and satisfying the relations (3.17), (3.18), (3.19), (3.20), (3.21).

These properties are not independent. In section 5, we will show that they are related with each other. Let us show now how one can reconstruct all the structures of the semiclassical field theory and check the properties of section 2. Let us consider the simplest example - the theory with classical action

$$I[\Phi] = \int dx[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)], \quad (3.22)$$

where $V(\Phi) \sim \frac{m^2}{2}\Phi^2$, $\Phi \to 0$.

Notice that the classical Poincare transformation is already constructed. Namely, $u_g J(x) = J(w_g x)$; $\overline{U}_g$ is X-independent. The 1-form $\underline{\omega}$ is of the form (3.3).

Say that $J \sim J'$ iff for some source J_+ with the support $suppJ_+ > suppJ$, $suppJ_+ > suppJ'$ the properties $J + J_+ \sim 0$, $J' + J_+ \sim 0$ are satisfied. This definition is equivalent to the following:

$$\overline{\Phi}(x|J) = \overline{\Phi}(x|J'), \quad x \underset{\sim}{>} suppJ, J' \quad \Leftrightarrow \quad J \sim J' \quad (3.23)$$

Say that $(\overline{S}_1, J_1) \sim (\overline{S}_2, J_2)$ iff

$$\overline{S}_1 + \overline{I}_{J_1+J_+} = \overline{S}_2 + \overline{I}_{J_2+J_+} \tag{3.24}$$

This definition does not depend on the particular choice of J_+. Consider the operator $\underline{V}(J_2 \leftarrow J_1)$ of the form

$$\underline{V}(J_2 \leftarrow J_1) = \underline{W}^+_{J_2+J_+} \underline{W}_{J_1+J_+}, \tag{3.25}$$

Formula (3.25) is also well-defined due to the Bogoliubov causality condition. Define the operatoe $\underline{\Phi}(x|J)$ as follows. If $x \underset{\sim}{>} suppJ$ then define $\underline{\Phi}(x|J)$ by relation (3.8). For the general case, choose a source $J' \sim J$ such that $suppJ' \underset{\sim}{<} x$ and set

$$\underline{\Phi}(x|J) = \underline{V}(J \leftarrow J') R(x|J') \underline{V}(J' \leftarrow J). \tag{3.26}$$

One chould check:

- whether eq.(3.26) is well-defined; this is a corollary of the identity

$$R(x|J_2)\underline{V}(J_2 \leftarrow J_1) = \underline{V}(J_2 \leftarrow J_1) R(x|J_1), \quad x \underset{\sim}{>} suppJ_{1,2}; \tag{3.27}$$

- property (2.7) (it is formulated as an axiom - eq.(3.1));
- relation (2.12) (it is reduced to eq.(3.6));
- eqs.(2.20) (they are corolloaries of the Poincare covariance of $\underline{W}$);
- eq.(2.22), which is of the form

$$\begin{gathered}\underline{V}(J_2 \leftarrow J_1)[-\delta \overline{S}_1 - \textstyle\int dx R(x|J_1)\delta J_1(x)] \\ = [-\delta \overline{S}_2 - \textstyle\int dx R(x|J_2)\delta J_2(x)] \underline{V}(J_2 \leftarrow J_1) \\ + ih\delta \underline{V}(J_2 \leftarrow J_1);\end{gathered} \tag{3.28}$$

- property (2.21) (it is a corollary of (3.27));
- relations (2.17), the first of them is a corollary of Poincare invariance, the second one can be presented as

$$ih\delta \underline{\Phi}(x|J) = [\underline{\Phi}(x|J); -\int dy R(y|J)\delta J(y)]; \tag{3.29}$$

- eq.(2.9) (it is a corollary of Poincare invariance of R and W).

Thus, to check the axioms of section 2, one should justify relations (3.27), (3.28), (3.29).

Eq.(3.28) is taken to the form

$$\delta\overline{S}_1 + \int dx \overline{\Phi}(x|J_1)\delta J_1(x) = \delta\overline{S}_2 + \int dx \overline{\Phi}(x|J_2)\delta J_2(x);$$

$$\begin{array}{c} -ih\delta\underline{W}_{J_2+J_+}\underline{W}^+_{J_2+J_+} + ih\delta\underline{W}_{J_1+J_+}\underline{W}^+_{J_1+J_+} = \\ -\underline{W}_{J_1+J_+}\int dx(R(x|J_1) - \overline{\Phi}(x|J_1))\delta J_1(x)\underline{W}^+_{J_1+J_+} \\ +\underline{W}_{J_2+J_+}\int dx(R(x|J_2) - \overline{\Phi}(x|J_2))\delta J_2(x)\underline{W}^+_{J_2+J_+} \end{array} \qquad (3.30)$$

here J_+ and $J_+ + \delta J_+$ are sources found from the relations $J_1 + J_+ \sim 0$, $J_2 + J_+ \sim 0$, $J_1 + \delta J_{1,2} + J_+ + \delta J_+ \sim 0$; the supports of the sources satisfy the conditions

$$supp\delta J_+ \underset{\sim}{>} suppJ_+ \underset{\sim}{>} suppJ \cup supp\delta J_1 \cup supp\delta J_2.$$

The Yang-Feldman relation (3.20) implies that

$$-ih\underline{W}^+_J\delta\underline{W}_J = \int dx[R(x|J) - \overline{\Phi}(x|J)]\delta J(x) \qquad (3.31)$$

for $J \sim 0$, $J + \delta J \sim 0$. Therefore, relation (3.30) is equivalent to the following one:

$$\begin{array}{c} 0 = -\underline{W}_{J_1+J_+}\int dx(R(x|J_1+J_+) - \overline{\Phi}(x|J_1+J_+))\delta J_+(x)\underline{W}^+_{J_1+J_+} \\ +\underline{W}_{J_2+J_+}\int dx(R(x|J_2+J_+) - \overline{\Phi}(x|J_2+J_+))\delta J_+(x)\underline{W}^+_{J_2+J_+}. \end{array}$$

This property is valid because of conditions on the support and boundary condition (3.21). Thus, the second relation (3.30) is checked, while the first one is a corollary of eq. (3.24).

Property (3.27) is a corollary of (3.28). Namely, one can choose a source δJ such that $supp\delta J \underset{\sim}{>} suppJ_{1,2}$.

Let us justify property (3.29). Consider the partial case $x \underset{\sim}{>} suppJ$, $x \underset{\sim}{>} supp\delta J$. Then relation

$$ih\delta R(x|J) = [R(x|J), -\int dy R(y|J)\delta J(y)], \quad x \underset{\sim}{>} suppJ, \delta J \qquad (3.32)$$

is a corollary of the commutation relation (3.6) and Bogoliubov causality property (3.5),

Consider now the general case. Choose sources J' and $\delta J'$ such that $J' \sim J$, $J' + \delta J' \sim J + \delta J$, $suppJ' \underset{\sim}{<} x$, $supp\delta J' \underset{\sim}{<} x$. Set

$$\underline{\Phi}(x|J) = \underline{V}(J \leftarrow J')R(x|J')\underline{V}(J' \leftarrow J).$$

Then eqs. (3.28) and (3.32) imply property (3.29).

4 The leading order of semiclassical expansion

4.1. In the previous section we have obtained the following properties of the operators:

$$\hat{\varphi}(x) \simeq \hat{\varphi}_0(x), \quad \underline{U}_g \simeq U_g,$$
$$\underline{W}[\overline{\Phi}] \simeq W[\overline{\Phi}], \quad R(x|J) \simeq \overline{\Phi}(x|J) + \sqrt{h} R^{(1)}(x|J)$$

Let us write them in the leading order of the semiclassical expansion. For $R^{(1)}$, one has:

$$U_{g_1 g_2} = U_{g_1} U_{g_2}, \quad U_{g^{-1}} R^{(1)}(x|u_g J) U_g = R^{(1)}(w_g x|J);$$

$$\begin{gathered} \left[R^{(1)}(x|J); R^{(1)}(y|J)\right] = -i(D^{ret}_{\overline{\Phi}_J}(x,y) - D^{ret}_{\overline{\Phi}_J}(y,x)) \equiv -iD_{\overline{\Phi}_J}(x,y); \\ R^{(1)}(x|J) = \hat{\varphi}_0(x), \quad x \lesssim supp J; \\ \frac{\delta R^{(1)}(x|J)}{\delta J(y)} = 0, \quad y \gtrsim x; \end{gathered} \tag{4.1}$$

Here $D^{ret}_{\overline{\Phi}}$ is a retarded Green function for the equation

$$(\partial_\mu \partial^\mu + V''(\overline{\Phi}(x)))\delta\Phi(x) = \delta J(x). \tag{4.2}$$

For $W[\overline{\Phi}]$, one has:

$$\begin{gathered} U_g W[\overline{\Phi}] U_{g^{-1}} = W[u_g \overline{\Phi}], \quad W^+[\overline{\Phi}] = (W[\overline{\Phi}])^{-1}; \\ \frac{\delta}{\delta \overline{\Phi}(y)} \left(W^+[\overline{\Phi}] \frac{\delta W[\overline{\Phi}]}{\delta \overline{\Phi}(x)} \right) = 0, \quad y \gtrsim x. \end{gathered} \tag{4.3}$$

For the case $J \sim 0$, the operator $R^{(1)}(x|J)$ should satisfy the following equations:

$$(\partial_\mu \partial^\mu + V''(\overline{\Phi}_J(x))) R^{(1)}(x|J) = 0; R^{(1)}(x|J) = W^+[\overline{\Phi}] \hat{\varphi}_0(x) W[\overline{\Phi}], \quad x \gtrsim supp \overline{\Phi}. \tag{4.4}$$

Let us construct the objects obeying relations (4.1), (4.3), (4.4). Because of the boundary conditions on $R^{(1)}$ at $x \lesssim supp J$, the field $\hat{\varphi}_0(x)$ satisfies the equations of motion and commutation relations for the free theory with the square mass $m^2 = V''(0)$. When one chooses the Fock representation of the canonical commutation relations, the oparator $\hat{\varphi}_0(x)$ coincides with the free scalar field with the mass m, while the operator U_g is a Poincare transformation in the free theory.

By $\hat{\phi}_\pm(x|\overline{\Phi})$ we denote the solution of the problem

$$\begin{gathered} (\partial_\mu \partial^\mu + V''(\overline{\Phi}(x)))\hat{\phi}_\pm(x|\overline{\Phi}) = 0, \\ \hat{\phi}_-|_{x \lesssim supp \overline{\Phi}} = \hat{\varphi}_0, \qquad \hat{\phi}_+|_{x \gtrsim supp \overline{\Phi}} = \hat{\varphi}_0. \end{gathered} \tag{4.5}$$

Then

$$R^{(1)}(x|J) = \hat{\phi}_-(x|\overline{\Phi}_J).$$

The properties of Poincare invariance and causality are evident; commutation relations (4.1) are checked as follows: one considers the difference between sides of equation $\Delta_{\overline{\Phi}}(x,y)$ and obtains the set of equations:

$$\begin{array}{c}(\partial^x_\mu \partial^\mu_x + V''(\overline{\Phi}(x)))\Delta_{\overline{\Phi}}(x,y) = 0,\\ (\partial^y_\mu \partial^\mu_y + V''(\overline{\Phi}(y)))\Delta_{\overline{\Phi}}(x,y) = 0,\\ \Delta_{\overline{\Phi}}(x,y) = 0, \qquad x, y \lesssim supp\overline{\Phi}.\end{array}$$

One then finds that $\Delta_{\overline{\Phi}}(x,y) = 0$.

The boundary condition (4.4) can be also presented as

$$\hat{\phi}_+(x|\overline{\Phi})W[\overline{\Phi}] = W[\overline{\Phi}]\hat{\phi}_-(x|\overline{\Phi}) \tag{4.6}$$

Consider whether there exists an unitary operator $W[\overline{\Phi}]$ satisfying the set of equations (4.3), (4.6). First, notice that it is defined up to a c-number phase factor $e^{i\gamma[\overline{\Phi}]}$. The phase should be Poincare invariant. Moreover, the causality property (4.3) imply the locality property $\frac{\delta^2\gamma[\overline{\Phi}]}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)} = 0, \quad x \neq y$. This phase factor is related to one-loop renormalization.

4.2. Investigate whether there exists an unitary operator $W[\overline{\Phi}]$ satisfying relation (4.6). The operators $\hat{\phi}_+(x|\overline{\Phi})$ and $\hat{\phi}_-(x|\overline{\Phi})$ obey the same commutation relation

$$[\hat{\phi}_\pm(x|\overline{\Phi}); \hat{\phi}_\pm(x|\overline{\Phi})] = \frac{1}{i}D_{\overline{\Phi}}(x,y),$$

It is checked analogously to (4.1). Therefore, the correspondence between operators $\hat{\phi}_\pm$ is a linear canonical transformation. To check that this transformation is unitary, one should justify the Berezin condition [14].

Denote

$$v(x) = V''(\overline{\Phi}(x)) - m^2; \tag{4.7}$$

then

$$\begin{array}{c}\hat{\phi}_+(x|\overline{\Phi}) = \hat{\varphi}_0(x) - \int dy D^{adv}_{\overline{\Phi}}(x,y)v(y)\hat{\varphi}_0(y),\\ \hat{\phi}_-(x|\overline{\Phi}) = \hat{\varphi}_0(x) - \int dy D^{ret}_{\overline{\Phi}}(x,y)v(y)\hat{\varphi}_0(y),\end{array}$$

where $D^{ret}_{\overline{\Phi}}(x,y)$ and $D^{adv}_{\overline{\Phi}}(x,y)$ are advanced and retarded Green functions for eq (4.2). For the case $x \gtrsim suppv$, relation (4.6) can be written as

$$W^+[\overline{\Phi}]\hat{\varphi}_0(x)W[\overline{\Phi}] = \hat{\varphi}_0(x) - \int dy D^{ret}_{\overline{\Phi}}(x,y)v(y)\hat{\varphi}_0(y), \quad x \gtrsim suppv. \tag{4.8}$$

The free-field creation and annihilation operators can be expressed via the field operators as

$$a^{\pm}_{\mathbf{p}} = \int_{x^0=const} d\mathbf{x} \frac{1}{(2\pi)^{d/2}} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left\{ \hat{\varphi}_0(x) \mp \frac{i}{\omega_{\mathbf{p}}} \frac{\partial}{\partial x^0} \hat{\varphi}_0(x) \right\} e^{\mp ipx}, \tag{4.9}$$

Therefore, eq.(4.8) is equivalent to the following relation:

$$W^+[\overline{\Phi}] a^{\pm}_{\mathbf{p}} W[\overline{\Phi}] = a^{\pm}_{\mathbf{p}} - \tfrac{1}{(2\pi)^{d/2}} \sqrt{\tfrac{\omega_{\mathbf{p}}}{2}} \int_{x^0=const} d\mathbf{x} dy v(y) \hat{\varphi}_0(y) e^{\mp ipx} \left\{ 1 \mp \tfrac{i}{\omega_{\mathbf{p}}} \tfrac{\partial}{\partial x^0} \right\} D^{ret}_{\overline{\Phi}}(x,y).$$

Therefore,

$$W^+[\overline{\Phi}] a^{\pm}_{\mathbf{p}} W[\overline{\Phi}] = a^{\pm}_{\mathbf{p}} + \int d\mathbf{k} [A^{\pm}_{\mathbf{pk}} a^{\pm}_{\mathbf{k}} + B^{\pm}_{\mathbf{pk}} a^{\mp}_{\mathbf{k}}], \tag{4.10}$$

where the c-number functions $A^{\pm}_{\mathbf{pk}}$ and $B^{\pm}_{\mathbf{pk}}$ rapidly damps at $\mathbf{p}, \mathbf{k} \to \infty$ as Fourier transformations of functions with compact support. According to Berezin condition, there exists $W[\overline{\Phi}]$ satisfying eq.(4.10) iff $B^{\pm}_{\mathbf{pk}} \in L^2$. For our case, this condition is satisfied; therefore, the operator W obeying eq.(4.6) exists.

Write

$$W[\overline{\Phi}] = c[\overline{\Phi}] W^0[\overline{\Phi}], \tag{4.11}$$

where $c[\overline{\Phi}]$ is a c-number factor; it absolute value is fixed due to unitarity, while $W^0[\overline{\Phi}]$ is an operator determined uniquely from eq. (4.6) and normalization condition

$$< 0 | W^0[\overline{\Phi}] | 0 >= 1. \tag{4.12}$$

Investigate the properties (4.3). The Poincare invariance relation can be rewritten as

$$c[u_g \overline{\Phi}] = c[\overline{\Phi}], \tag{4.13}$$

since the operator $U_{g^{-1}} W[u_g \overline{\Phi}] U_g$ satisfis the commutation relations (4.6) and coincides with $W[\overline{\Phi}]$ up to a c-multiplier.

To investigate unitarity and causality properties, introduce the current operator

$$\hat{\rho}(x|\overline{\Phi}) \equiv \frac{1}{i} W^+[\overline{\Phi}] \frac{\delta W[\overline{\Phi}]}{\delta \overline{\Phi}(x)}. \tag{4.14}$$

If W is an unitary operator then the current is Hermitian. When one mmultiplies the operator $W[\overline{\Phi}]$ by a c-number factor, the operator $\hat{\rho}(x|\overline{\Phi})$ is increased by a c-number; therefore, the multiplier $c[\overline{\Phi}]$ satisfies the unitarity condition iff

$$< 0 | \hat{\rho}(x | \overline{\Phi} | 0 > = < 0 | \hat{\rho}^+(x | \overline{\Phi} | 0 > . \tag{4.15}$$

The causality condition means that

$$\frac{\delta\hat{\rho}(x|\overline{\Phi})}{\delta\overline{\Phi}(y)} = 0, \qquad y \gtrsim x. \tag{4.16}$$

4.3. Let us obtain an explicit form of the operator $\hat{\rho}(x|\overline{\Phi})$. Differentiating relation (4.6) with respect to $\overline{\Phi}$, we obtain that

$$W^+[\overline{\Phi}]\tfrac{\delta\hat{\phi}_+(x|\overline{\Phi})}{\delta\overline{\Phi}(y)}W[\overline{\Phi}] + i\hat{\phi}_-(x|\overline{\Phi})\hat{\rho}(y|\overline{\Phi}) = i\hat{\rho}(y|\overline{\Phi})\hat{\phi}_-(x|\overline{\Phi}) + \tfrac{\delta\hat{\phi}_-(x|\overline{\Phi})}{\delta\overline{\Phi}(y)}. \tag{4.17}$$

Variating eq.(4.5), we find:

$$\begin{array}{c} \delta\hat{\phi}_-(x|\overline{\Phi}) = -\int D^{ret}_{\overline{\Phi}}(x,y)V'''(\overline{\Phi}(y))\hat{\phi}_-(y|\overline{\Phi})\delta\overline{\Phi}(y)dy; \\ \delta\hat{\phi}_+(x|\overline{\Phi}) = -\int D^{adv}_{\overline{\Phi}}(x,y)V'''(\overline{\Phi}(y))\hat{\phi}_+(y|\overline{\Phi})\delta\overline{\Phi}(y)dy \end{array} \tag{4.18}$$

Therefore,

$$i[\hat{\rho}(y|\overline{\Phi});\hat{\phi}_-(x|\overline{\Phi})] = D_{\overline{\Phi}}(x,y)V'''(\overline{\Phi}(y))\hat{\phi}_-(y|\overline{\Phi}).-\int D^{ret}_{\overline{\Phi}}(x,y) \tag{4.19}$$

Thus, the operator $\hat{\rho}(y|\overline{\Phi})$ is reconstructed up to an additive constant. It can be expressed via the normal products [6, 8]. As usual, define the positive- and negative-frequency (at $-\infty$) parts $\hat{\phi}^{\pm}_-(y|\overline{\Phi})$ of the field $\hat{\phi}_-(y|\overline{\Phi})$ due to the relations

$$(\partial_\mu\partial^\mu + V''(\overline{\Phi}(x)))\hat{\phi}^{\pm}_-(x|\overline{\Phi}) = 0, \quad \hat{\phi}^{\pm}_-|_{x \lesssim supp\overline{\Phi}} = \hat{\varphi}^{\pm}_0. \tag{4.20}$$

Therefore,

$$\hat{\rho}(y|\overline{\Phi}) = --\frac{1}{2}V'''(\overline{\Phi}(y))\left[:\hat{\phi}^2_-(y|\overline{\Phi}): + \alpha(y|\overline{\Phi})\right], \tag{4.21}$$

where $\alpha(y|\overline{\Phi})$ is a multiplicator by a real number. Consider the vacuum averaging values of left-hand and right-hand sides of the relations

$$iW[\overline{\Phi}]\hat{\rho}(x|\overline{\Phi}) = \frac{\delta W[\overline{\Phi}]}{\delta\overline{\Phi}(x)}.$$

We obtain the relation for finding $c[\overline{\Phi}]$:

$$\begin{array}{c} \frac{\delta c[\overline{\Phi}]}{\delta\overline{\Phi}(x)} = i<0|W[\overline{\Phi}]\hat{\rho}(x|\overline{\Phi})|0>= \\ -\frac{i}{2}V'''(\overline{\Phi}(x))<0|W[\overline{\Phi}]\left[\hat{\phi}^+_-(x|\overline{\Phi})\hat{\phi}^+_-(x|\overline{\Phi}) + \alpha(y|\overline{\Phi})\right]|0>. \end{array} \tag{4.22}$$

Investigate the commutation rules between operators $W[\overline{\Phi}]$ and $\hat{\phi}_-^{\pm}$. First, write

$$\hat{\phi}_-(x|\overline{\Phi}) = \hat{\varphi}_0(x) - \int dy D_0^{ret}(x,y)v(y)\hat{\phi}_-(y|\overline{\Phi});$$

therefore, for $x \underset{\sim}{>} suppv$ one finds from (4.6) that

$$W^+[\overline{\Phi}]\hat{\varphi}_0(x)W[\overline{\Phi}] = \hat{\varphi}_0(x) - \int dy D_0^{ret}(x,y)v(y)\hat{\phi}_-(y|\overline{\Phi}), \quad x \underset{\sim}{>} suppv. \tag{4.23}$$

Notice that the positive-frequency part of the function $D_0^{ret}(x,y)$ coincides with $D_0^+(x,y)$ at $x \underset{\sim}{>} suppv$. Therefore, it follows from eq.(4.23) that

$$\begin{gathered} W^+[\overline{\Phi}]\hat{\varphi}_0^+(x)W[\overline{\Phi}] = \hat{\varphi}_0^+(x) - \int dy D_0^+(x,y)v(y)\hat{\phi}_-(y|\overline{\Phi}) \\ = \hat{\phi}_-^+(x|\overline{\Phi}) + \int dy (D_0^{ret}(x,y) - D_0^+(x,y))v(y)\hat{\phi}_-^+(y|\overline{\Phi}) - \int dy D_0^+(x,y)v(y)\hat{\phi}_-^-(y|\overline{\Phi}). \end{gathered} \tag{4.24}$$

Notice also that

$$D_0^{ret} - D_0^+ = D_0^c.$$

Investigate the corrolaries of relation (4.24).

Denote by $\check{A}$ the operator with the kernel $A(x,y)$. Multiply eq.(4.24) by $W[\overline{\Phi}]$ and use relation $< 0|\hat{\varphi}_0^+(x) = 0$. We obtain that

$$0 =< 0|W[\overline{\Phi}]\hat{\phi}_-^+(x|\overline{\Phi}) - < 0|W[\overline{\Phi}]\left\{(\check{1} + \check{D}_0^c\check{v})^{-1}\check{D}_0^+\check{v}\check{\phi}_-\right\}(x|\overline{\Phi})$$

Here $\check{v}$ is the operator of multiplication by $v(x)$ with the kernel $v(x,y) = v(x)\delta(x-y)$.

Relation (4.22) is taken to the form

$$\frac{\delta c[\overline{\Phi}]}{\delta\overline{\Phi}(x)} = -\frac{i}{2}c[\overline{\Phi}]V'''(\overline{\Phi}(x))\left[\frac{1}{i}\{(\check{1} + \check{D}_0^c\check{v})^{-1}\check{D}_0^+\check{v}\check{D}_{\overline{\Phi}}^-\}(x,x) + \alpha(x|\overline{\Phi})\right]. \tag{4.25}$$

Here $D_{\overline{\Phi}}^-$ is a commutation function for the operators $\hat{\phi}_-^-$ $\hat{\phi}_-^+$:

$$[\hat{\phi}_-^-(\xi_1|\overline{\Phi}); \hat{\phi}_-^+(\xi_2|\overline{\Phi})] = \frac{1}{i}D_{\overline{\Phi}}^-(\xi_1,\xi_2).$$

Eq.(4.25) can be also rewritten as

$$\alpha\{x|v\} = 2i\frac{\delta \ln c[v]}{\delta v(x)} + i\{(\check{1} + \check{D}_0^c\check{v})^{-1}\check{D}_0^+\check{v}\check{D}_{\overline{\Phi}}^-\}(x,x). \tag{4.26}$$

The following identity

$$(\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^+\check{v}\check{D}_{\overline{\Phi}}^- = -\check{D}_{\overline{\Phi}}^- + (\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^c - (\check{1}+\check{D}_0^{adv}\check{v})^{-1}\check{D}_0^{adv}.$$

is checked by a direct calculation. The last term satisfies the property $\{(\check{1}+\check{D}_0^{adv}\check{v})^{-1}\check{D}_0^{adv}\}(x,x)=0$; therefore,

$$\alpha\{x|v\} = 2i\frac{\delta \ln c[v]}{\delta v(x)} + i\{-\check{D}_{\overline{\Phi}}^- + (\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^c\}(x,x). \tag{4.27}$$

Since $ReD_{\overline{\Phi}}^-(x,x)=0$, $\frac{\delta D_{\overline{\Phi}}^-(x,x)}{\delta v(y)}=0$ at $y \gtrsim x$, the Poincare invariance, unitarity and causality properties can be presented as

$$\begin{gathered} c[u_g v] = c[v]; \\ Re\left\{\frac{\delta c[v]}{\delta v(x)} + \frac{1}{2}[(\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^c](x,x)\right\} = 0; \\ \frac{\delta}{\delta v(y)}\left\{\frac{\delta c[v]}{\delta v(x)} + \frac{1}{2}[(\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^c](x,x)\right\} = 0, \quad y \gtrsim x. \end{gathered} \tag{4.28}$$

Formally, the solution of system (4.28) has the form

$$c[v] = (det(\check{1}+\check{D}_0^c\check{v}))^{-1/2}e^{i\gamma[v]}.$$

Here $\gamma[v]$ is an arbitrary real local functional satisfying the condition $\frac{\delta^2\gamma[v]}{\delta v(x)\delta v(y)}=0$ as $x\neq y$. Relation (4.28) may be viewed as a definition of the determinant. Although the quantity $[(\check{1}+\check{D}_0^c\check{v})^{-1}\check{D}_0^c](x,x)$ diverges, its real part and variation derivative with respect to $v(y)$ at $y\neq x$ are finite.

Eq.(4.26) implies the important property:

$$\frac{\delta\alpha\{x|v\}}{\delta v(y)} - \frac{\delta\alpha\{y|v\}}{\delta v(x)} = -i\left(\frac{\delta D_{\overline{\Phi}}^-(x,x)}{\delta v(y)} - \frac{\delta D_{\overline{\Phi}}^-(y,y)}{\delta v(x)}\right) = i[(D_{\overline{\Phi}}^-(x,y))^2 - (D_{\overline{\Phi}}^-(y,x))^2] \tag{4.29}$$

This will be used further.

Relation (4.29) can be also obtained from the identity

$$[\hat{\rho}(x|\overline{\Phi};\hat{\rho}(y|\overline{\Phi}] = -i\left(\frac{\delta\hat{\rho}(x|\overline{\Phi})}{\delta\overline{\Phi}(y)} - \frac{\delta\hat{\rho}(y|\overline{\Phi})}{\delta\overline{\Phi}(x)}\right). \tag{4.30}$$

It follows from eq.(4.29) and causality condition that

$$\frac{\delta\alpha\{x|v\}}{\delta v(y)} = i[(D_{\overline{\Phi}}^-(x,y))^2 - (D_{\overline{\Phi}}^-(y,x))^2]\theta(x^0-y^0). \tag{4.31}$$

For $x=y$, one should clarify relation (4.31). This depends on the choice of one-loop counterterm.

5 Semiclassical S-matrix

5.1. Let us analyze the semiclassical perturbation theory. It is necessary to fix the representation. Otherwise, the next order of perturbation theory for the operators $\underline{U}_g$, $\hat{\varphi}(x)$, $R(x|J)$, $\underline{W}[\overline{\Phi}]$ will be defined non-uniquely due to choice of representation.

In qunatum field theory, one usually uses the asymptotic in-representation: one supposes that the particles become free at $t \to -\infty$. Therefore, the state space can be identified with the free Fock space of in-particles. For this case, the Heisenberg field $\hat{\varphi}(x)$ weakly tends as $t \to \pm\infty$ to the asymptotic free field:

$$\hat{\varphi}(x) \sim_{x^0 \to -\infty} \hat{\varphi}_{in}(x) \equiv \hat{\varphi}_0(x); \quad \hat{\varphi}(x) \sim_{x^0 \to +\infty} \hat{\varphi}_{out}(x).$$

The S-matrix (it will be denoted as $\hat{\Sigma}_0$) is an unitary transformation between asymptotic fields:

$$\hat{\varphi}_{out}(x) = \hat{\Sigma}_0^+ \hat{\varphi}_{in}(x) \hat{\Sigma}_0 = \hat{\Sigma}_0^+ \hat{\varphi}_0(x) \hat{\Sigma}_0. \tag{5.1}$$

Denote

$$\hat{\Sigma}_{\overline{\Phi}} \equiv \hat{\Sigma}_0 \underline{W}[\overline{\Phi}]. \tag{5.2}$$

Making use of these notations and assumptions, write down the formulas for perturbation theory:

$$\underline{U}_{g_1 g_2} = \underline{U}_{g_1} \underline{U}_{g_2}, \quad \underline{U}_{g^{-1}} \hat{\varphi}_0(x) \underline{U}_g = \hat{\varphi}_0(w_g x); \tag{5.3}$$

$$\begin{gathered} \underline{U}_g \hat{\Sigma}_{\overline{\Phi}} \underline{U}_{g^{-1}} = \hat{\Sigma}_{u_g \overline{\Phi}}, \quad \hat{\Sigma}^+_{\overline{\Phi}} = \hat{\Sigma}^{-1}_{\overline{\Phi}}; \\ \frac{\delta}{\delta \overline{\Phi}(y)} \left(\hat{\Sigma}^+_{\overline{\Phi}} \frac{\delta \hat{\Sigma}^+_{\overline{\Phi}}}{\delta \overline{\Phi}(x)} \right) = 0, \qquad y \gtrsim x. \end{gathered} \tag{5.4}$$

As $J \sim 0$, the R-function $R(x|J) \equiv R\{x|\overline{\Phi}\}$ should satisfy the Yang-Feldman relation:

$$(\partial_\mu \partial^\mu + V''(\overline{\Phi}(x)))(R\{x|\overline{\Phi}\} - \overline{\Phi}(x)) = -ih \hat{\Sigma}^+_{\overline{\Phi}} \frac{\delta \hat{\Sigma}^+_{\overline{\Phi}}}{\delta \overline{\Phi}(x)} \tag{5.5}$$

and boundary conditions:

$$\begin{gathered} R\{x|\overline{\Phi}\} \sim_{x^0 \to -\infty} \sqrt{h} \hat{\varphi}_0(x); \\ R\{x|\overline{\Phi}\} \sim_{x^0 \to +\infty} \underline{W}^+[\overline{\Phi}] \sqrt{h} \hat{\varphi}_{out}(x) \underline{W}[\overline{\Phi}] = \hat{\Sigma}^+_{\overline{\Phi}} \sqrt{h} \hat{\varphi}_0(x) \hat{\Sigma}_{\overline{\Phi}}. \end{gathered} \tag{5.6}$$

Therefore, the R-function can be presented in two forms, which are corollaries of boundary conditions (5.6):

$$\begin{gathered} R\{x|\overline{\Phi}\} - \overline{\Phi}(x) = \hat{\phi}_-(x|\overline{\Phi}) \sqrt{h} - ih \int dy D^{ret}_{\overline{\Phi}}(x,y) \hat{\Sigma}^+_{\overline{\Phi}} \frac{\delta \hat{\Sigma}_{\overline{\Phi}}}{\delta \overline{\Phi}(y)}; \\ R\{x|\overline{\Phi}\} - \overline{\Phi}(x) = \hat{\Sigma}^+_{\overline{\Phi}} \left[\hat{\phi}_+(x|\overline{\Phi}) \sqrt{h} - ih \int dy D^{adv}_{\overline{\Phi}}(x,y) \frac{\delta \hat{\Sigma}_{\overline{\Phi}}}{\delta \overline{\Phi}(y)} \hat{\Sigma}^+_{\overline{\Phi}} \right] \hat{\Sigma}_{\overline{\Phi}}. \end{gathered} \tag{5.7}$$

We obtain the following important identity:

$$\hat{\phi}_+(x|\overline{\Phi})\hat{\Sigma}_{\overline{\Phi}} - \hat{\Sigma}_{\overline{\Phi}}\hat{\phi}_-(x|\overline{\Phi}) = -i\sqrt{h}\int dy D_{\overline{\Phi}}(x,y)\frac{\delta\hat{\Sigma}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}, \tag{5.8}$$

since the commutation function has the form:

$$D_{\overline{\Phi}}(x,y) = D^{ret}_{\overline{\Phi}}(x,y) - D^{adv}_{\overline{\Phi}}(x,y).$$

5.2. Let $\hat{\varphi}_0(x)$ be a scalar free field of the mass m, $\underline{U}_g$ be the Poincare transformation in this theory, Let us show that the S-matrix properties (5.4) and (5.8) imply all other relations of section 3.

At $J \sim 0$, define $R(x|J) \equiv R\{x|\overline{\Phi}\}$ from the first formula (5.7). Because of the Bogoliubov causality property (5.4), the $R(x|J)$-function depends on $J(y)$ at $y < x$ only. Therefore, the definition can be extended to the case $J \not\sim 0$: set $R(x|J) \equiv R(x|J+J_+)$, where $suppJ_+ \gtrsim suppJ$, $J + J_+ \sim 0$. The causality property (3.5) remains valid. The Poincare invariance property (3.4) is a corollary of covariance of relation (5.7).

Set $\hat{\varphi}(x)\sqrt{h} \equiv R(x|0)$ and check relation (3.21). Rewrite it (at $x \gtrsim supp\overline{\Phi}$) as

$$\hat{\Sigma}_0\hat{\varphi}(x)\sqrt{h}\hat{\Sigma}_0^+ = \hat{\Sigma}_{\overline{\Phi}}\left[\hat{\phi}_-(x|\overline{\Phi})\sqrt{h} - ih\int dy D^{ret}_{\overline{\Phi}}(x,y)\hat{\Sigma}^+_{\overline{\Phi}}\frac{\delta\hat{\Sigma}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}\right]\hat{\Sigma}^+_{\overline{\Phi}}$$

or (because of (5.8))

$$\hat{\Sigma}_0\hat{\varphi}(x)\sqrt{h}\hat{\Sigma}_0^+ = \hat{\phi}_+(x|\overline{\Phi})\sqrt{h} - ih\int dy D^{adv}_{\overline{\Phi}}(x,y)\frac{\delta\hat{\Sigma}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}\hat{\Sigma}^+_{\overline{\Phi}} \tag{5.9}$$

It follows from the causality relation that $\frac{\delta\hat{\Sigma}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}\hat{\Sigma}^+_{\overline{\Phi}}$ depends only on $\overline{\Phi}(z)$ for $z > y$. Since $\overline{\Phi}(y) = 0$ at $y > x$, property (5.9) is taken to the form

$$\hat{\Sigma}_0\hat{\varphi}(x)\sqrt{h}\hat{\Sigma}_0^+ = \hat{\phi}_+(x|0)\sqrt{h} - ih\int dy D^{adv}_0(x,y)\frac{\delta\hat{\Sigma}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}|_{\overline{\Phi}=0}\hat{\Sigma}^+_0 \tag{5.10}$$

However, the property (5.10) is a corollary of (5.7) and (5.8) for $\overline{\Phi} = 0$.

The commutation relation (3.6) is checked by a direct computation. Its sketch is as follows.

1. One writes relation (3.6) in the more convenient form:

$$\begin{gathered}[\textstyle\int dx R(x|J)\delta_1 J(x), \int dy R(y|J)\delta_2 J(y)] = \\ -ih\textstyle\int dx[\delta_2 R(x|J)\delta_1 J(x) - \delta_1 R(x|J)\delta_2 J(x)];\end{gathered} \tag{5.11}$$

for $R(x|J)$, one uses the first relation (5.7):

$$\begin{gathered} R(x|J) = \overline{\Phi}(x) + \sqrt{h} r_-(x|\overline{\Phi}); \\ r_-(x|\overline{\Phi}) = \hat{\phi}_-(x|\overline{\Phi}) + \sqrt{h} \int dy D^{ret}_{\overline{\Phi}}(x,y) \hat{j}(y|\overline{\Phi}); \\ \hat{j}(y|\overline{\Phi}) = \frac{1}{i} \hat{\Sigma}^+_{\overline{\Phi}} \frac{\delta \hat{\Sigma}^+_{\overline{\Phi}}}{\delta \overline{\Phi}(y)}. \end{gathered} \tag{5.12}$$

2. One takes the right-hand side of (5.11) to the form:

$$\begin{gathered} -ih\{\int dx[\delta_2 \overline{\Phi}(x) \delta_1 J(x) - \delta_1 \overline{\Phi}(x) \delta_2 J(x)] \\ -\sqrt{h} \int dx dz D^{ret}_{\overline{\Phi}}(x,z) V'''(\overline{\Phi}(z)) r_-(z|\overline{\Phi}) [\delta_2 \overline{\Phi}(z) \delta_1 J(x) - \delta_1 \overline{\Phi}(z) \delta_2 J(x)] \\ +h \int dx dy D^{ret}_{\overline{\Phi}}(x,y) [\delta_2 \hat{j}(y|\overline{\Phi}) \delta_1 J(x) - \delta_1 \hat{j}(y|\overline{\Phi}) \delta_2 J(x)]\} \end{gathered} \tag{5.13}$$

3. One variates the relation

$$\hat{\Sigma}^+_{\overline{\Phi}} \hat{\phi}_+(x|\overline{\Phi}) \hat{\Sigma}_{\overline{\Phi}} - \hat{\phi}_-(x|\overline{\Phi}) = \sqrt{h} \int dy D_{\overline{\Phi}}(x,y) \hat{j}(y|\overline{\Phi})$$

with respect to $\overline{\Phi}(z)$. After all computations, one obtains:

$$-i[\hat{j}(z|\overline{\Phi}), \hat{\phi}_-(x|\overline{\Phi})] = \sqrt{h} \int dy D_{\overline{\Phi}}(x,y) \frac{\delta \hat{j}(z|\overline{\Phi})}{\delta \overline{\Phi}(y)} - D_{\overline{\Phi}}(x,y) V'''(\overline{\Phi}(z)) \hat{r}_-(z|\overline{\Phi}). \tag{5.14}$$

Further, it follows from unitarity that

$$[\hat{j}(\xi|\overline{\Phi}); \hat{j}(z|\overline{\Phi})] = -i \left(\frac{\delta \hat{j}(\xi|\overline{\Phi})}{\delta \overline{\Phi}(z)} - \frac{\delta \hat{j}(z|\overline{\Phi})}{\delta \overline{\Phi}(\xi)} \right) \tag{5.15}$$

4. Making use of the commutation relations (5.14) and (5.15), one takes the left-hand side of eq.(5.11) to the form (5.13).

5.3. Relation (5.8) can be viewed as a basis of the semiclassical perturbation theory. It is convenient to consider the substitution

$$\hat{\Sigma}_{\overline{\Phi}} = W[\overline{\Phi}] \tilde{\Sigma}_{\overline{\Phi}};$$

then formula (5.8) will be taken to the form

$$[\hat{\phi}_-(x|\overline{\Phi}); \tilde{\Sigma}_{\overline{\Phi}}] = \sqrt{h} \int dy D_{\overline{\Phi}}(x,y) \left\{ \hat{\rho}(y|\overline{\Phi}) \tilde{\Sigma}_{\overline{\Phi}} - i \frac{\delta \tilde{\Sigma}_{\overline{\Phi}}}{\delta \overline{\Phi}(y)} \right\}, \tag{5.16}$$

where $\hat\rho(y|\overline{\Phi})$ has the form (4.21). Expand $\tilde\Sigma_{\overline{\Phi}}$ into an asymptotic series in $\sqrt{h}$:

$$\tilde\Sigma_{\overline{\Phi}} = 1 + \sqrt{h}\tilde\Sigma^{(1)}_{\overline{\Phi}} + h\tilde\Sigma^{(2)}_{\overline{\Phi}} + ...$$

Therefore, for the k-th order of the perturbation theory, one obtains the following recursive relations:

$$[\hat\phi_-(x|\overline{\Phi}); \tilde\Sigma^{(k)}_{\overline{\Phi}}] = \sqrt{h}\int dy D_{\overline{\Phi}}(x,y)\left\{\hat\rho(y|\overline{\Phi})\tilde\Sigma^{(k-1)}_{\overline{\Phi}} - i\frac{\delta\tilde\Sigma^{(k-1)}_{\overline{\Phi}}}{\delta\overline{\Phi}(y)}\right\}. \tag{5.17}$$

Investigate whether there exists $\tilde\Sigma^{(k)}_{\overline{\Phi}}$ satisfying eq.(5.17). It is convenient to use symbolic calculus based on the notion of a normal symbol [8]. One expands the operators into a series containing normal products of the fields $\hat\phi_-(x|\overline{\Phi})$:

$$\tilde\Sigma^{(k)}_{\overline{\Phi}} = \sum_m \int dx_1...dx_m \tilde\Sigma^{(k)}_{\overline{\Phi},m}(x_1,...,x_m) : \hat\phi_-(x_1|\overline{\Phi})...\hat\phi_-(x_m|\overline{\Phi}) : \tag{5.18}$$

By $\tilde S^{(k)}_{\overline{\Phi}}[\phi_-(\cdot)]$ we denote the normal symbol:

$$\tilde S^{(k)}_{\overline{\Phi}}[\phi_-(\cdot)] = \sum_m \int dx_1...dx_m \tilde\Sigma^{(k)}_{\overline{\Phi},m}(x_1,...,x_m)\phi_-(x_1)...\phi_-(x_m).$$

One also has

$$\tilde\Sigma^{(k)}_{\overline{\Phi}} =: \tilde S^{(k)}_{\overline{\Phi}}[\phi_-(\cdot|\overline{\Phi})] :$$

Introduce the following notations. By $A * B$ we denote the normal symbol of the propduct of operators $\hat A =: A[\hat\phi_-(\cdot|\overline{\Phi})]:$ and $\hat B =: B[\hat\phi_-(\cdot|\overline{\Phi})]:$. It is equal to

$$(A*B)[\phi_-(\cdot)] = A[\hat\phi_-(\cdot) + \frac{1}{i}\int dy D^-_{\overline{\Phi}}(\cdot,y)\frac{\delta}{\delta\phi_-(y)}]B[\phi_-(\cdot)], \tag{5.19}$$

by $\frac{DA}{D\overline{\Phi}(x)}$ let us denote the normal symbol of the operator $\frac{\delta\hat A}{\delta\overline{\Phi}(x)}$. Its explicit form is

$$\frac{DA}{D\overline{\Phi}(x)}[\phi_-(\cdot)] = \frac{\delta A}{\delta\overline{\Phi}(x)}[\phi_-(\cdot)] - \int d\xi D^{ret}_{\overline{\Phi}}(\xi,x)V'''(\overline{\Phi})\phi_-(x)\frac{\delta A}{\delta\phi_-(\xi)}[\phi_-(\cdot)] \tag{5.20}$$

Since the normal symbol of the commutator $[\hat\phi_-(x|\overline{\Phi}); \hat A]$ has the form $\int dy \frac{1}{i} D_{\overline{\Phi}}(x,y)\frac{\delta A}{\delta\phi_-(y)}$, property (5.17) is taken to the form

$$\frac{\delta\tilde S^{(k)}_{\overline{\Phi}}}{\delta\phi_-(y)} = i\rho(y|\overline{\Phi}) * \tilde S^{(k-1)}_{\overline{\Phi}} + \frac{D\tilde S^{(k-1)}_{\overline{\Phi}}}{D\overline{\Phi}(y)}, \tag{5.21}$$

where
$$\rho(y|\overline{\Phi}) = \rho(y|\overline{\Phi},\phi_-) = -\frac{1}{2}V'''(\overline{\Phi}(y))[\phi_-^2(y) + \alpha(y|\overline{\Phi})]-$$
is a normal symbol of the operator $\hat{\rho}(y|\overline{\Phi})$.

Relation (5.21) can be solved and $\tilde{S}^{(k)}[\phi]$ can be found under the following condition
$$[i\rho(y|\overline{\Phi}) * + \frac{D}{D\overline{\Phi}(y)}; i\rho(z|\overline{\Phi}) * + \frac{D}{D\overline{\Phi}(z)}] = 0. \tag{5.22}$$
It follows from (5.20) that $[\frac{D}{D\overline{\Phi}(y)};\frac{D}{D\overline{\Phi}(z)}] = 0$, so that relation (5.22) is taken to the form:
$$i\left(\frac{D\rho(z|\overline{\Phi})}{D\overline{\Phi}(y)} - \frac{D\rho(y|\overline{\Phi})}{D\overline{\Phi}(z)}\right) = \rho(y|\overline{\Phi}) * \rho(z|\overline{\Phi}) - \rho(z|\overline{\Phi}) * \rho(y|\overline{\Phi}). \tag{5.23}$$
Formula (5.23) is another form of eq.(4.30). It is checked in analogous way, It implies that there exist $\tilde{S}^{(k)}_{\overline{\Phi}}$ and $\tilde{\Sigma}^{(k)}_{\overline{\Phi}}$ satisfying eqs. (5.21) and (5.17).

The operator $\tilde{\Sigma}^{(k)}_{\overline{\Phi}}$ is found form eq.(5.17) up to a multiplier by a c-number $a^{(k)}_{\overline{\Phi}}$; therefore,
$$\tilde{\Sigma}^{(k)}_{\overline{\Phi}} = \tilde{\Sigma}^{(k)}_{\overline{\Phi}0} + a^{(k)}_{\overline{\Phi}}, \tag{5.24}$$
here $\tilde{\Sigma}^{(k)}_{\overline{\Phi}0}$ is uniquely determined from the normalization condition
$$< 0|\tilde{\Sigma}^{(k)}|0 >= 0.$$

Investigate now the properties of Poincare invariance, unitarity and causality. Notice that the operator $U_g\tilde{\Sigma}^{(k)}_{u_g\overline{\Phi}}U_g^{-1}$ satisfies eq. (5.17) and coincides up to an additional c-number constant with $\tilde{\Sigma}^{(k)}_{\overline{\Phi}}$. Therefore, the property of Poincare invariance is satisfied if
$$a^{(k)}_{u_g\overline{\Phi}} = a^{(k)}_{\overline{\Phi}}. \tag{5.25}$$
To check the unitarity property, notice that the operator $\tilde{\Sigma}_{\overline{\Phi}}$ obeying relation (5.16) also satisfies the condition
$$[\hat{\phi}_-(x|\overline{\Phi}); \tilde{\Sigma}^+_{\overline{\Phi}}\tilde{\Sigma}_{\overline{\Phi}}] = \sqrt{h}\int dy D_{\overline{\Phi}}(x,y)(-i)\frac{\delta}{\delta\overline{\Phi}(y)}(\tilde{\Sigma}^+_{\overline{\Phi}}\tilde{\Sigma}_{\overline{\Phi}}). \tag{5.26}$$
By $(\tilde{\Sigma}^+_{\overline{\Phi}}\tilde{\Sigma}_{\overline{\Phi}})_k$ we denote the k-th order of perturbative expansion of the operator $(\tilde{\Sigma}^+_{\overline{\Phi}}\tilde{\Sigma}_{\overline{\Phi}})$ into a series in $\sqrt{h}$. Then the commutator $[\hat{\phi}_-(x|\overline{\Phi}); (\tilde{\Sigma}^+_{\overline{\Phi}}\tilde{\Sigma}_{\overline{\Phi}})_k]$ will be expressed from

(5.26) via $(\tilde\Sigma^+_{\overline\Phi}\tilde\Sigma_{\overline\Phi})_{k-1}$. It will vanish if the unitarity property is satisfied in the previous order of the perturbation theory. Therefore, the operator $(\tilde\Sigma^+_{\overline\Phi}\tilde\Sigma_{\overline\Phi})_k$ is a multiplicator by a c-number. It vanishes if

$$0 =< 0|(\tilde\Sigma^+_{\overline\Phi}\tilde\Sigma_{\overline\Phi})_k|0 >\equiv 2Re a^{(k)}_{\overline\Phi} + \sum_{s=1}^{k-1} < 0|\tilde\Sigma^{(s)+}_{\overline\Phi}\tilde\Sigma^{(k-s)}_{\overline\Phi}|0 > . \tag{5.27}$$

Relation (5.27) fixes the real part of $a^{(k)}_{\overline\Phi}$ uniquely.

Investigate now the causality property. Making use of notations (5.12), consider the commutation relation (5.14) at $x \underset{\sim}{<} supp\overline\Phi \;\; x^0 < z^0$:

$$[\hat\varphi_0(x); \hat j(z|\overline\Phi)] = -i\sqrt{h}\int dy D_{\overline\Phi}(x,y)\frac{\delta \hat j(x|\overline\Phi)}{\delta\overline\Phi(y)} + iD_{\overline\Phi}(x,z)V'''(\overline\Phi(z))\hat r_-(z|\overline\Phi).$$

Therefore, for the variational derivative $\frac{\delta \hat j(z|\overline\Phi)}{\delta\overline\Phi(\xi)} \;\; \xi^0 > z^0$ one has:

$$[\varphi_0(x); \frac{\delta \hat j^{(k)}(z|\overline\Phi)}{\delta\overline\Phi(\xi)}] = -i\int dy \frac{\delta}{\delta\overline\Phi(\xi)}\left(D_{\overline\Phi}(x,y)\frac{\delta \hat j^{(k-1)}(z|\overline\Phi)}{\delta\overline\Phi(y)},\right)$$

where $\hat j^{(k)}(z|\overline\Phi)$ is the k-th order of perturbation theory for $\hat j(z|\overline\Phi)$. Therefore, the Bogoliubov causality condition in the $(k-1)$-th order implies that the k-th order for the operator $\frac{\delta \hat j^{(k)}(z|\overline\Phi)}{\delta\overline\Phi(\xi)}$ at $z^0 < \xi^0$ will be a c-number multiplier. Therefore, the causality property is satisfied in the k-th order iff

$$\frac{\delta}{\delta\overline\Phi(\xi)} < 0|\hat j^{(k)}(z|\overline\Phi)|0 >= 0, \quad \xi \underset{\sim}{>} z. \tag{5.28}$$

Let us calculate the vacuum average of the operator $\hat j^{(k)}$. Present eq.(5.12) as

$$\hat j(x|\overline\Phi) = \tilde\Sigma^+_{\overline\Phi}\hat\rho(x|\overline\Phi)\tilde\Sigma_{\overline\Phi} + \frac{1}{i}\tilde\Sigma^+_{\overline\Phi}\frac{\delta\tilde\Sigma_{\overline\Phi}}{\delta\overline\Phi(x)}. \tag{5.29}$$

Therefore,

$$\frac{1}{i} < 0|\frac{\delta\tilde\Sigma_{\overline\Phi}}{\delta\overline\Phi(x)}|0 >=< 0|\tilde\Sigma_{\overline\Phi}\hat j(x|\overline\Phi)|0 > - < 0|\hat\rho(x|\overline\Phi)\tilde\Sigma_{\overline\Phi}|0 >, \tag{5.30}$$

and $\hat\rho(x|\overline\Phi) = \hat j^{(0)}(x|\overline\Phi)$. Consider the k-th order of the perturbation theory for (5.30). We come to the following relations:

$$< 0|\hat j^{(k)}(x|\overline\Phi)|0 >= \frac{1}{i}\frac{\delta a^{(k)}_{\overline\Phi}}{\delta\overline\Phi(x)} - \textstyle\sum_{s=1}^{k-1} < 0|\tilde\Sigma^{(s)}_{\overline\Phi}\hat j^{(k-s)}(x|\overline\Phi)|0 > - < 0|[\tilde\Sigma^{(k)}_{\overline\Phi}; \hat\rho(x|\overline\Phi)]|0 > . \tag{5.31}$$

Properties (5.27) and (5.31) allows us to obtain the conditions on imaginary part of $a^{(k)}_{\overline{\Phi}}$:

$$\frac{1}{i}\frac{\delta^2 a^{(k)}_{\overline{\Phi}}}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)} = \frac{\delta}{\delta\overline{\Phi}(y)}\left[\sum_{s=1}^{k-1} <0|\tilde{\Sigma}^{(s)}_{\overline{\Phi}}\hat{j}^{(k-s)}(x|\overline{\Phi})|0> + <0|[\tilde{\Sigma}^{(k)}_{\overline{\Phi}};\hat{\rho}(x|\overline{\Phi})]|0>\right], \quad y \underset{\sim}{>} x. \tag{5.32}$$

Thus, for each order of the perturbation theory one should construct the operator $\tilde{\Sigma}^{(k)}_{\overline{\Phi}}$ from eq. (5.17) (or (5.21)) and choose a c-number multiplier $a^{(k)}_{\overline{\Phi}}$ from the Poincare invariance condition (5.25), unitarity property (5.27) and causality relation (5.32). Then the properties (5.4) of the operator $\hat{\Sigma}_{\overline{\Phi}}$ will be satisfied within the perturbation framework.

The c-number functional $a^{(k)}_{\overline{\Phi}}$ is defined up to a purely imaginary Poincare invariant functional $i\gamma^{(k)}_{\overline{\Phi}}$ satisfying the locality property $\frac{\delta^2\gamma^{(k)}_{\overline{\Phi}}}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)} = 0$, $x \neq y$. The non-uniqueness is related with the possibility of one-loop renormalization of the Lagrangian,

5.4. To illustrate the obtained relations, consider the lower orders of perturbation theory.

The zero order gives the relation $\tilde{S}^{(0)}_{\overline{\Phi}} = 1$; the first order for relation (5.21) implies that

$$\frac{\delta\tilde{S}^{(1)}_{\overline{\Phi}}}{\delta\phi_-(y)} = -\frac{i}{2}V'''(\overline{\Phi}(y))[\phi_-^2(y) + \alpha(y|\overline{\Phi})];$$

therefore,

$$\tilde{S}^{(1)}_{\overline{\Phi}} = -i\int dx V'''(\overline{\Phi}(x))\left[\frac{1}{6}\phi_-^3(x) + \frac{1}{2}\alpha(x|\overline{\Phi})\phi_-(x)\right] + a^{(1)}_{\overline{\Phi}}$$

The unitarity and causality properties (5.27) and (5.32) will be written as

$$Re a^{(1)}_{\overline{\Phi}} = 0, \quad \frac{1}{i}\frac{\delta^2 a^{(1)}_{\overline{\Phi}}}{\delta\overline{\Phi}(x)\delta\overline{\Phi}(y)} = \frac{\delta}{\delta\overline{\Phi}(y)} <0|[\tilde{\Sigma}^{(1)}_{\overline{\Phi}};\hat{\rho}(x|\overline{\Phi})]|0>.$$

Therefore, it is possible to choose $a^{(1)}_{\overline{\Phi}} = 0$.

The second order of perturbation theory gives us the following relations

$$\begin{gathered}\rho(y|\overline{\Phi}) * \tilde{S}^{(1)}_{\overline{\Phi}} = \\ \tfrac{i}{2}V'''(\overline{\Phi}(y))(\phi_-^2(y) + \alpha(y|\overline{\Phi}))\int dx V'''(\overline{\Phi}(x))\left[\tfrac{1}{6}\phi_-^3(x) + \tfrac{1}{2}\alpha(x|\overline{\Phi})\phi_-(x)\right] \\ +\tfrac{1}{2}V'''(\overline{\Phi}(y))\phi_-(y)\int dx V'''(\overline{\Phi}(x))D^-_{\overline{\Phi}}(y,x)[\phi_-^2(x) + \alpha(x|\overline{\Phi})] \\ -\tfrac{i}{2}V'''(\overline{\Phi}(y))\int dx[D^-_{\overline{\Phi}}(y,x)]^2 V'''(\overline{\Phi}(x))\phi_-(x);\end{gathered}$$

$$
\begin{array}{c}
\frac{D\tilde{S}^{(1)}}{D\overline{\Phi}(y)} = -iV^{(IV)}(\overline{\Phi}(y))\left[\frac{1}{6}\phi_-^3(y)+\frac{1}{2}\alpha(y|\overline{\Phi})\phi_-(y)\right] \\
-\frac{i}{2}\int dx V'''(\overline{\Phi}(x))\phi_-(x)\frac{\delta\alpha(x|v)}{\delta v(y)}V'''(\overline{\Phi}(y)) \\
-\int dx D^{ret}_{\overline{\Phi}}(x,y)V'''(\overline{\Phi}(y))\phi_-(y)(-\frac{i}{2}V'''(\overline{\Phi}(x)))[\phi_-^2(x)+\alpha(x|\overline{\Phi})].
\end{array}
$$

Combining the results, we obtain from eq. (5.21) the following realtion for $\tilde{S}^{(2)}$:

$$
\begin{array}{c}
\frac{\delta\tilde{S}^{(2)}}{\delta\phi_-(y)} = -iV^{(IV)}(\overline{\Phi}(y))\left[\frac{1}{6}\phi_-^3(y)+\frac{1}{2}\alpha(y|\overline{\Phi})\phi_-(y)\right] \\
-\frac{1}{2}V'''(\overline{\Phi}(y))(\phi_-^2(y)+\alpha(y|\overline{\Phi}))\int dx V'''(\overline{\Phi}(x))\left[\frac{1}{6}\phi_-^3(x)+\frac{1}{2}\alpha(x|\overline{\Phi})\phi_-(x)\right] \\
+\frac{i}{2}V'''(\overline{\Phi}(y))\phi_-(y)\int dx V'''(\overline{\Phi}(x))[D^-_{\overline{\Phi}}+D^{adv}_{\overline{\Phi}}](y,x)(\phi_-^2(x)+\alpha(x|\overline{\Phi})) \\
+\frac{1}{2}V'''(\overline{\Phi}(y))\int dx V'''(\overline{\Phi}(x))\phi_-(x)\left\{[D^-_{\overline{\Phi}}(y,x)]^2+\frac{1}{i}\frac{\delta\alpha\{x|v\}}{\delta v(y)}\right\}
\end{array}
$$

For the simplicity, introduce the notations:

$$
\begin{array}{c}
D^c_{\overline{\Phi}} \equiv D^-_{\overline{\Phi}}+D^{adv}_{\overline{\Phi}}; \\
(D^c_{\overline{\Phi}}(y,x))^2_R \equiv [D^-_{\overline{\Phi}}(y,x)]^2+\frac{1}{i}\frac{\delta\alpha\{x|v\}}{\delta v(y)}.
\end{array} \tag{5.33}
$$

Formally, it follows from eq.(4.31) that

$$
\begin{array}{c}
D^c_{\overline{\Phi}}(y,x) = \theta(x^0-y^0)D^-_{\overline{\Phi}}(x,y)\theta(y^0-x^0)D^-_{\overline{\Phi}}(y,x); \\
(D^c_{\overline{\Phi}}(y,x))^2_R = \theta(x^0-y^0)(D^-_{\overline{\Phi}}(x,y))^2\theta(y^0-x^0)(D^-_{\overline{\Phi}}(y,x))^2 = (D^c_{\overline{\Phi}}(y,x))^2;
\end{array} \tag{5.34}
$$

However, formulas (5.34) should be clarified at $x=y$ due to divergences.

One has:

$$
\begin{array}{c}
\tilde{S}^{(2)} = -i\int dx V^{(IV)}(\overline{\Phi}(x))\left[\frac{1}{24}\phi_-^4(x)+\frac{1}{4}\alpha(x|\overline{\Phi})\phi_-^2\right] \\
-\frac{1}{4}\left\{\int dx V'''(\overline{\Phi}(x))\left[\frac{1}{6}\phi_-^3(x)+\frac{1}{2}\alpha(x|\overline{\Phi})\phi_-(x)\right]\right\}^2 \\
+\frac{i}{4}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(y))D^c_{\overline{\Phi}}(x,y) \\
\times\left([\phi_-^2(x)+\alpha(x|\overline{\Phi})][\phi_-^2(y)+\alpha(y|\overline{\Phi})]-\alpha(x|\overline{\Phi})\alpha(y|\overline{\Phi})\right) \\
+\frac{1}{2}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(y))(D^c_{\overline{\Phi}}(x,y))^2_R\phi_-(x)\phi_-(y)+a^{(2)}_{\overline{\Phi}}.
\end{array}
$$

The c-number $a^{(2)}_{\overline{\Phi}}$ satisfies the properties of Poincare invariance, unitarity and causality. It happens that one can look for $a^{(2)}_{\overline{\Phi}}$ in the following form:

$$
\begin{array}{c}
a^{(2)}_{\overline{\Phi}} = -\frac{i}{8}\int dx V^{(IV)}(\overline{\Phi}(x))(\alpha(x|\overline{\Phi}))^2 \\
-\frac{1}{8}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(y))\alpha(x|\overline{\Phi})\alpha(y|\overline{\Phi})\frac{1}{i}D^c_{\overline{\Phi}}(x,y) \\
-\frac{1}{12}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(y))(\frac{1}{i}D^c_{\overline{\Phi}}(x,y))^3_R
\end{array}
$$

where $(D^c_{\overline{\Phi}})^3_R$ is a renormalized cub of the function $D^c_{\overline{\Phi}}$.

The unitarity condition (5.27) can be written as

$$Rea^{(2)}_{\overline{\Phi}} = -\frac{1}{2} < 0|\tilde{\Sigma}^{(1)+}_{\overline{\Phi}} \tilde{\Sigma}^{(1)}_{\overline{\Phi}}|0 >;$$

it leads to the relation

$$Re(\frac{1}{i} D^c_{\overline{\Phi}}(x,y))^3_R = \frac{1}{2}\left[(\frac{1}{i} D^-_{\overline{\Phi}}(x,y))^3 + (\frac{1}{i} D^-_{\overline{\Phi}}(y,x))^3 \right], \tag{5.35}$$

and

$$Re(\frac{1}{i} D^c_{\overline{\Phi}}(x,y)) = \frac{1}{2}\left[(\frac{1}{i} D^-_{\overline{\Phi}}(x,y)) + (\frac{1}{i} D^-_{\overline{\Phi}}(y,x)) \right], \tag{5.36}$$

Relation (5.36) is obviously satisfied; property (5.35) is an important condition on $(D^c_{\overline{\Phi}})^3_R$.

Consider the vacuum average value of $\hat{j}^{(2)}$ according to formula (5.31):

$$< 0|\hat{j}^{(2)}(\xi|\overline{\Phi})|0 >= \frac{1}{i}\frac{\delta a^{(2)}_{\overline{\Phi}}}{\delta\overline{\Phi}(\xi)} - < 0|\tilde{\Sigma}^{(1)}_{\overline{\Phi}} \hat{j}^{(1)}(\xi|\overline{\Phi})|0 > - < 0|[\tilde{\Sigma}^{(2)}_{\overline{\Phi}}; \hat{\rho}(\xi|\overline{\Phi})]|0 > .$$

The normal symbol

$$\hat{j}^{(1)}(\xi|\overline{\Phi}) = \frac{1}{i}\frac{\delta\tilde{\Sigma}^{(1)}_{\overline{\Phi}}}{\delta\overline{\Phi}(\xi)} + [\hat{\rho}(\xi); \tilde{\Sigma}^{(1)}_{\overline{\Phi}}]$$

has the form

$$\begin{gathered} j^{(1)}(\xi|\overline{\Phi},\phi_-) = -V^{(IV)}(\overline{\Phi}(\xi))\left[\tfrac{1}{6}\phi^3_-(\xi) + \tfrac{1}{2}\alpha(\xi|\overline{\Phi})\phi_-(\xi)\right] \\ +\tfrac{1}{2}V'''(\overline{\Phi}(\xi))\phi_-(\xi)\int dx V'''(\overline{\Phi}(x)) D^{ret}_{\overline{\Phi}}(\xi,x)[\phi^2_-(x) + \alpha(x|\overline{\Phi})] \\ -\tfrac{1}{2}V'''(\overline{\Phi}(\xi))\int dx V'''(\overline{\Phi}(x))\phi_-(x)\tfrac{\delta\alpha\{\xi|v\}}{\delta v(x)}. \end{gathered}$$

Therefore,

$$\begin{gathered} < 0|\hat{j}^{(2)}(\xi|\overline{\Phi})|0 >= -\tfrac{1}{8}V^{(V)}(\overline{\Phi}(\xi))(\alpha(\xi|\overline{\Phi})^2 - \tfrac{1}{4}\int dx V^{(IV)}(\overline{\Phi}(x))V'''(\overline{\Phi}(\xi))\alpha(x|\overline{\Phi})\tfrac{\delta\alpha\{\xi|v\}}{\delta v(x)} \\ -\tfrac{1}{6}\int dy V^{(IV)}(\overline{\Phi}(\xi))V'''(\overline{\Phi}(y))[(D^c_{\overline{\Phi}}(\xi,y))^3_R - (D^-_{\overline{\Phi}}(y,\xi))^3] \\ +\tfrac{1}{4}\int dy V^{(IV)}(\overline{\Phi}(\xi))V'''(\overline{\Phi}(y))\alpha\{\xi|v\}\alpha\{y|v\}[(D^c_{\overline{\Phi}}(\xi,y)) - (D^-_{\overline{\Phi}}(y,\xi))] \\ +\tfrac{1}{4}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(\xi))V'''(\overline{\Phi}(y))G^1_v(\xi,x,y) \\ +\tfrac{1}{4}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(\xi))V'''(\overline{\Phi}(y))\alpha\{x|v\}\tfrac{\delta\alpha\{\xi|v\}}{\delta v(y)}[D^c_{\overline{\Phi}}(x,y) - D^-_{\overline{\Phi}}(x,y)] \\ +\tfrac{1}{8}\int dxdy V'''(\overline{\Phi}(x))V'''(\overline{\Phi}(\xi))V'''(\overline{\Phi}(y))\alpha\{x|v\}\alpha\{y|v\}G^2_v(\xi,x,y). \end{gathered}$$

Here

$$G^1_v(\xi,x,y) = -\frac{1}{3}\frac{\delta}{\delta v(\xi)}(D^c_{\Phi}(x,y))^3_R - D^{ret}_{\Phi}(\xi,y)D^-_{\Phi}(x,\xi)(D^-_{\Phi}(x,y))^2$$
$$-D^{ret}_{\Phi}(\xi,x)D^-_{\Phi}(y,\xi)(D^-_{\Phi}(y,x))^2 - (D^c_{\Phi}(x,y))^2_R(D^-_{\Phi}(y,\xi)D^-_{\Phi}(x,\xi) - D^-_{\Phi}(\xi,y)D^-_{\Phi}(\xi,x)),$$

$$G^2(\xi,x,y) = \frac{\delta D^c_{\Phi}(x,y)}{\delta v(\xi)} + D^{ret}_{\Phi}(\xi,y)D^-_{\Phi}(x,\xi) + D^{ret}_{\Phi}(\xi,x)D^-_{\Phi}(y,\xi)$$
$$+D^-_{\Phi}(x,\xi)D^-_{\Phi}(y,\xi) - D^-_{\Phi}(\xi,x)D^-_{\Phi}(\xi,y) = -D^{ret}_{\Phi}(\xi,x)D^{ret}_{\Phi}(\xi,y).$$

The causality condition (5.28) is satisfied, provided that

$$\begin{aligned} G^1(\xi,x,y) = 0, \quad \xi \lesssim x \quad \xi \lesssim y; \\ (D^c_{\Phi}(\xi,y))^3_R = (D^c_{\Phi}(\xi,y))^3 \quad \xi \lesssim y. \end{aligned} \tag{5.37}$$

Notice that formally $(D^c_{\Phi}(\xi,y))^3_R = (D^c_{\Phi}(\xi,y))^3$; moreover,

$$G^1(\xi,x,y) = D^{ret}_{\Phi}(\xi,x)D^{ret}_{\Phi}(\xi,y)(D^c_{\Phi}(x,y))^2_R$$
$$-D^{ret}_{\Phi}(\xi,y)D^-_{\Phi}(x,\xi)[(D^-_{\Phi}(x,y))^2 - (D^c_{\Phi}(x,y))^2]$$
$$-D^{ret}_{\Phi}(\xi,x)D^-_{\Phi}(y,\xi)[(D^-_{\Phi}(y,x))^2 - (D^c_{\Phi}(x,y))^2]$$

and conditions (5.37) are satisifed. However, the function $(D^c_{\Phi})^3_R$ contains the divergences which are eliminated by the renormalization procedure, so that relations (5.36) and (5.37) are important for renormalization.

6 Unstable particles

In the previous section, we have supposed that the usual assumptions of the S-matrix theory are satisfied. On the other hand, when one investigate the bound states and unstable particles, the S-matrix conception leads to difficulties. Let us show that semiclassical perturbation theory can be developed even for the unstable particle case. Consider a simple example. Let the action of the model be

$$I[\Phi,X] = \int dx \left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}\partial_\mu X\partial^\mu X - \frac{m^2}{2}\Phi^2 - \frac{M^2}{2}X^2 + \Phi^2 X\right].$$

By $\hat{\varphi}(x)$, $\hat{\chi}(x)$ we denote the corresponding quantum fields. It is well-known that for the case $M > 2m$, one X-particle can decay into two Φ-particles. Investigate this process.

By $a^{\pm}_{\mathbf{p}}$ we denote the creation and annihilation operators for the free Φ-particles. Let $b^{\pm}_{\mathbf{p}}$ be creation and annihilation operators for the free X-particles. Then the zeroth order of perturbation theory gives the relation

$$\hat{\chi}_0(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d\mathbf{p}}{\sqrt{2\Omega_{\mathbf{p}}}} [b^+_{\mathbf{p}} e^{i\Omega_{\mathbf{p}} t - i\mathbf{p}x} + b^-_{\mathbf{p}} e^{-i\Omega_{\mathbf{p}} t + i\mathbf{p}x}]; \quad \Omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2};$$
$$\hat{\varphi}_0(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} [a^+_{\mathbf{p}} e^{i\omega_{\mathbf{p}} t - i\mathbf{p}x} + a^-_{\mathbf{p}} e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}x}]; \quad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}.$$

Investigate the operator of the form $\hat{\chi}(x)$ in the firt order of perturbation theory:

$$\hat{\chi} = \hat{\chi}_0 + \sqrt{h}\hat{\chi}_1 + ...$$

For zero claical fields, the Yang-Feldman relation (3.20) for the quantum field $\hat{\chi}(x)$ takes the form:

$$(\partial_\mu \partial^\mu + M^2)\chi(x) = -i\sqrt{h} \frac{\delta \underline{W}[\overline{\Phi}, \overline{X}]}{\delta \overline{X}(x)} |_{\overline{\Phi}, \overline{X} = 0};$$

therefore,

$$(\partial_\mu \partial^\mu + M^2)\chi_1(x) = -i \frac{\delta W[\overline{\Phi}, \overline{X}]}{\delta \overline{X}(x)} |_{\overline{\Phi}, \overline{X} = 0}. \tag{6.1}$$

Analogously to eq.(4.21), we obtain that the right-hand side of relation (6.1) has the form

$$-iW^+ \frac{\delta W}{\delta \overline{X}(x)} =$$
$$: \left[\frac{1}{2} \frac{\partial^3 V}{\partial \Phi^2 \partial X} \hat{\phi}^2_-(x|\overline{\Phi}, \overline{X}) + \frac{\partial^3 V}{\partial \Phi \partial X^2} \hat{\phi}_-(x|\overline{\Phi}, \overline{X}) \hat{\chi}_-(x|\overline{\Phi}, \overline{X}) + \frac{1}{2} \frac{\partial^3 V}{\partial X^3} \hat{\chi}^2_-(x|\overline{\Phi}, \overline{X}) \right] : +\alpha(x|\overline{\Phi}, \overline{X})$$

As $\overline{\Phi}, \overline{X} = 0$,

$$-i \frac{\delta W}{\delta \overline{X}(x)} |_{\overline{\Phi}, \overline{X} = 0} = - : \hat{\varphi}^2_0(x) : + \alpha_0(x).$$

In particular, the positive-frequency part $\hat{\chi}^{++}_1(x)$ with two creation operators a^+ obeys the following equation:

$$(\partial_\mu \partial^\mu + M^2) \hat{\chi}^{++}_1(x) = -(\hat{\varphi}_0(x))^2 =$$
$$-\frac{1}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} a^+_{\mathbf{k}_1} a^+_{\mathbf{k}_2} e^{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})t - i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}}. \tag{6.2}$$

The solution of eq.(6.2) is non-unique, since we have not fixed the representation of canonical commutation relations. Let us fix a representation. First, require that the momentum operator has the form:

$$\mathbf{P} = \int d\mathbf{k}\mathbf{k} (a^+_{\mathbf{k}} a^-_{\mathbf{k}} + b^+_{\mathbf{k}} b^-_{\mathbf{k}}),$$

Then the operator $\hat{\chi}_1^{++}$ will have the form:

$$\hat{\chi}_1^{++}(x) = -\frac{1}{(2\pi)^d}\int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}}\frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} a^+_{\mathbf{k}_1} a^+_{\mathbf{k}_2}$$
$$\times \left[\frac{1}{\Omega^2_{\mathbf{k}_1+\mathbf{k}_2} - (\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})^2} e^{i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})t - i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}} + \tilde{g}_{\mathbf{k}_1\mathbf{k}_2} e^{i\Omega_{\mathbf{k}_1+\mathbf{k}_2}t - i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}}\right]$$

where $\tilde{g}_{\mathbf{k}_1\mathbf{k}_2}$ is an arbitrary function. It depends on the particular choice of the representation. The integrand should not contain the poles. Therefore,

$$\tilde{g}_{\mathbf{k}_1\mathbf{k}_2} = -\frac{1}{\Omega^2_{\mathbf{k}_1+\mathbf{k}_2} - (\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})^2} + g_{\mathbf{k}_1\mathbf{k}_2},$$

where the function $g_{\mathbf{k}_1\mathbf{k}_2}$ has no poles. Therefore,

$$\hat{\chi}_1^{++}(x) = -\frac{1}{(2\pi)^d}\int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}}\frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} a^+_{\mathbf{k}_1} a^+_{\mathbf{k}_2} e^{i\Omega_{\mathbf{k}_1+\mathbf{k}_2}t - i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}}$$
$$\times \left[\frac{1}{\Omega^2_{\mathbf{k}_1+\mathbf{k}_2} - (\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})^2}(e^{i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\Omega_{\mathbf{k}_1+\mathbf{k}_2})t} - 1) + g_{\mathbf{k}_1\mathbf{k}_2}\right] \quad (6.3)$$

Let us calculate now the decay rate..Let the initial condition have the form:

$$\Psi_0 \equiv \int d\mathbf{x}dt\hat{\chi}(\mathbf{x},t)\alpha(\mathbf{x},t)|0> \quad (6.4)$$

In the zeroth order of perturbation theory, one has

$$\Psi_0 \simeq \int d\mathbf{p}\frac{\tilde{\alpha}(\mathbf{p},\Omega_\mathbf{p})}{\sqrt{2\Omega_\mathbf{p}}} b^+_\mathbf{p}|0>,$$

with

$$\tilde{\alpha}(\mathbf{p},\varepsilon) \equiv \frac{1}{(2\pi)^{d/2}}\int d\mathbf{x}d\tau e^{i\varepsilon\tau - i\mathbf{p}\mathbf{x}},$$

Therefore, the state contains one X-particle with the wave function

$$\psi_0(\mathbf{p}) = \frac{\tilde{\alpha}(\mathbf{p},\Omega_\mathbf{p})}{\sqrt{2\Omega_\mathbf{p}}}. \quad (6.5)$$

Evaluate the probability amplitude that at time t there will be two Φ-particles with momenta $\mathbf{k}_1$ $\mathbf{k}_2$:

$$\psi_{\mathbf{k}_1\mathbf{k}_2}(t) = \frac{1}{\sqrt{2}} < 0|a^-_{\mathbf{k}_1}a^-_{\mathbf{k}_2}e^{-iHt}|\Psi_0 >= \frac{1}{\sqrt{2}} < 0|a^-_{\mathbf{k}_1}a^-_{\mathbf{k}_2}\int d\mathbf{x}d\tau\hat{\chi}(\mathbf{x},\tau - t)\alpha(\mathbf{x},\tau)|0 >;$$
$$(6.6)$$

Here the property $e^{-iHt}\hat{\chi}(\mathbf{x},\tau)e^{iHt} = \hat{\chi}(\mathbf{x},\tau-t)$ is used. The only nontrivial contribution to the matrix element (6.6) is given by the components $\hat{\chi}_1^{++}$; therefore, it follows from eq.(6.3) that:

$$\psi_{\mathbf{k}_1\mathbf{k}_2} = -\frac{\sqrt{2h}}{(2\pi)^{d/2}}\frac{1}{\sqrt{2\omega_{\mathbf{k}_1}}}\frac{1}{\sqrt{2\omega_{\mathbf{k}_2}}}e^{-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})t}$$
$$\times\left[\frac{1}{\Omega^2_{\mathbf{k}_1+\mathbf{k}_2}-(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})^2}\tilde{\alpha}(\mathbf{k}_1+\mathbf{k}_2,\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})+\right.$$
$$\left.\tilde{\alpha}(\mathbf{k}_1+\mathbf{k}_2,\Omega_{\mathbf{k}_1+\mathbf{k}_2})e^{i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\Omega_{\mathbf{k}_1+\mathbf{k}_2})t}\left(g_{\mathbf{k}_1\mathbf{k}_2}-\frac{1}{\Omega^2_{\mathbf{k}_1+\mathbf{k}_2}-(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})^2}\right)\right]$$

Consider the rate of transition

$$\Gamma_{\mathbf{k}_1\mathbf{k}_2} = \frac{1}{t}|\psi_{\mathbf{k}_1\mathbf{k}_2}(t)|^2$$

Let $t\to\infty$. Making use of formula $\left|\frac{e^{i\alpha t}-1}{\alpha}\right|^2\frac{1}{t}\simeq 2\pi\delta(\alpha)$, one finds that

$$\Gamma_{\mathbf{k}_1\mathbf{k}_2} \simeq \frac{2h}{(2\pi)^d}\frac{1}{2\omega_{\mathbf{k}_1}}\frac{1}{2\omega_{\mathbf{k}_2}}\frac{1}{(2\Omega_{\mathbf{k}_1+\mathbf{k}_2})^2}2\pi\delta(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\Omega_{\mathbf{k}_1+\mathbf{k}_2})|\tilde{\alpha}(\mathbf{k}_1+\mathbf{k}_2,\Omega_{\mathbf{k}_1+\mathbf{k}_2})|^2.$$

Substitute the initial wave function:

$$\Gamma_{\mathbf{k}_1\mathbf{k}_2} \simeq \frac{2h}{(2\pi)^d}\frac{1}{2\omega_{\mathbf{k}_1}}\frac{1}{2\omega_{\mathbf{k}_2}}\frac{1}{2\Omega_{\mathbf{k}_1+\mathbf{k}_2}}2\pi\delta(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\Omega_{\mathbf{k}_1+\mathbf{k}_2})|\psi_0(\mathbf{k}_1+\mathbf{k}_2)|^2$$

Let the initial momentum of the particle be equal to $\mathbf{P}$; then $|\psi_0(\mathbf{p})|^2 = \delta(\mathbf{p}-\mathbf{P})$, and the standard formula for the decay rate is reproduced.

Thus, the semiclassical approach allows us to reproduce the quantum field theory results for decay rates.

7 Conclusions

Thus, semiclassical perturbation field theory can be constructed analogously to the axiomatic quantum field theory. The axioms of Poincare covariance, unitarity and Bogoliubov causality are formulated analogously; however, the correspondence principle between quantum and classical theories gives new relations. Therefore, the scattering matrix on the external background field is calculated in each order of perturbation expansion up to *a c-number* local term, not up to quasilocal *operator.*

The semiclassical perturbation theory may be generalized to the case of unstable particles.

The generalization to Fermi fields is not difficult: one should only substitute commuting variables by Grassmannian variables in a standard way. The case of constrained systems and gauge fields is non-trivial; the author is going to discuss it in further publications.

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